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Induced Representations of Crossed Products by Coactions

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Let δ: $A \to \tilde{M}(A \otimes C_r^*(G))$ be a coaction of a locally compact group G on a C^*-algebra A. Then for any closed normal amenable subgroup H of G we define a coaction $\delta| : A \to \tilde{M}(A \otimes C_r^*(G/H))$ of G/H on A. We present dense $*$-subalgebras of the crossed products $A \times_\delta G$ and $A \times_{\delta|} (G/H)$ and use these to induce representations of $A \times_{\delta|} (G/H)$ to representations of $A \times_\delta G$. We then formulate an imprimitivity theorem for the induction process; that is, we classify the induced representations. We also investigate the continuity of the induction process and are able to prove Rieffel's conjecture that the dual action to a coaction is proper and saturated. © 1991 Academic Press, Inc.

In 1898, in his analysis of the representation theory of finite groups, Frobenius [6] introduced a method for constructing representations (called induced representations) of a group from those of its subgroups. Some 40 years later, Nakayama [18] and Weil [28] generalised his construction to include compact groups, and in [29], Wigner was able to extend these ideas and thus describe the representations of the (non-compact) inhomogeneous Lorentz group. Building on this Mackey [15–17] developed a theory of induced representations for arbitrary separable locally compact groups and proved an imprimitivity theorem characterising those representations that can be obtained from his construction.

Much of the theory of representations of groups is subsumed in the theory of $*$-representations of involutive Banach algebras. In particular, Rieffel [21] has shown that the theory of induced representations of groups can be reformulated in the context of C^*-algebras. Fell [3, 5] has also given a closely related theory of induced representations in the context of Banach $*$-algebraic bundles, but we shall adopt Rieffel's approach. Rieffel's work states that if we can find dense subalgebras $\mathscr{E}$ and $\mathscr{F}$ of C^*-algebras E and F, respectively, and a bimodule X which carries a left $\mathscr{E}$-action, a right $\mathscr{F}$-action and an $\mathscr{F}$-valued inner product satisfying

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various conditions, then we can construct representations of E from those of F (see Section 1 for details). His work also includes a very general imprimitivity theorem involving the algebra, $K(X)$, of "compact operators" of the $\mathscr{F}$-rigged module X.

Earlier work by Takesaki [25] established a theory of induced representations for crossed products by actions (covariance algebras) which, as shown by Green [8], can be exhibited within Rieffel's framework. Green's version of Takesaki's work states that if α is an action of a locally compact group G on a C^*-algebra A and H is a closed subgroup of G, then $C_c(G, A)$ $(=X)$ is a left $C_c(G, A)$ $(=\mathscr{E})$, right $C_c(H, A)$ $(=\mathscr{F})$ bimodule with a $C_c(H, A)$-valued inner product which satisfies Rieffel's conditions. Hence one can construct representations of $A \times_\alpha G$ $(=E)$ from representations of $A \times_\alpha H$ $(=F)$; that is, one has an induction process. By identifying the algebra, $K(X)$, of compact operators as the crossed product $(A \otimes C_0(G/H)) \times_{\alpha \otimes \tau} G$, where τ is the left translation action of G on $C_0(G/H)$, Green is also able to reformulate Takesaki's characterisation of the induced representations (imprimitivity theorem) in Rieffel's setting.

It is the intention of this paper to present an analogous induction process and imprimitivity theorem for representations of crossed products by coactions. More precisely, if δ is a coaction of G on A, H is a closed normal amenable subgroup of G, and $\delta|$ is the "restriction" of δ to a coaction of G/H on A (see Section 2), then using Rieffel's theory we will establish a procedure for constructing representations of $A \times_\delta G$ from those of $A \times_{\delta|} (G/H)$ and will be able to characterise those representations which can be obtained in this way. This will require finding suitable candidates for $\mathscr{E}$, $\mathscr{F}$, and X, and the determination of the algebra $K(X)$ in terms of $A \times_\delta G$ and other known quantities.

In [24], Rieffel introduces "proper" actions of groups on C^*-algebras. These actions have been defined so as to be a generalisation of proper actions of groups on locally compact spaces and are closely related to the integrable actions of [1, 2]. As a by-product of our construction we are able to verify Rieffel's conjecture that the dual action to a coaction is proper and saturated.

Gootman and Lazar [7] have presented a different notion of induction for crossed products by coactions: they define the representation of $A \times_\delta G$ induced from the representation ϱ of A $(=A \times_{\delta|} (G/G))$ to be $((\varrho \otimes i) \circ \delta) \times (1 \otimes M_G)$, where M_G is the representation of $C_b(G)$ on $L^2(G)$ by multiplication operators. We show (Proposition 21) that this notion is a special case ours (when $H = G$). We also generalise their result [7, Theorem 3.8] on the continuity of the induction process.

In Section 1 we gather a little background material. In Section 2 we introduce the restriction $\delta|$ of a coaction δ and show that given a faithful representation, π, of A on some Hilbert space $\mathscr{H}$, the crossed product

$A \times_{\delta|} (G/H)$ can be faithfully represented on $\mathscr{H} \otimes L^2(G)$. We will generally choose to work with this copy of $A \times_{\delta|} (G/H)$ in $B(\mathscr{H} \otimes L^2(G))$, which we will denote $A \times_\delta (G/H)$, rather than $A \times_{\delta|} (G/H)$ itself. We begin the presentation of the main results in Section 3, where we show that if φ averages elements of $C_c(G)$ over H-cosets, then the set $\mathscr{D}_H$ of norm limits of sequences $(x_j)_{j=1}^{\infty}$ in $B(\mathscr{H} \otimes L^2(G))$ of the form

$$x_j = \sum_{i=1}^{n_j} \{\pi \otimes i\}(\delta(\delta_u(a_{ij})))(1 \otimes M_G(\varphi(f_{ij}))),$$

where the a_{ij} are in A, the f_{ij} are elements of $C_c(G)$, all of which have support in some fixed compact subset of G, and where u is some fixed compactly supported element of the Fourier algebra, is a dense $*$-subalgebra of $A \times_\delta (G/H)$. We are now in a position to present our candidates for $\mathscr{E}$, $\mathscr{F}$, and X. If 1 denotes the trivial subgroup of G, then $\mathscr{D}_1$ is a dense $*$-subalgebra of $A \times_\delta (G/1) = A \times_\delta G$ and is our choice for $\mathscr{E}$. Our choice for $\mathscr{F}$ is $\mathscr{D}_H$. In Section 4 we show that $\mathscr{D}_1$, our candidate for X, can be equipped with a $\mathscr{D}_H$-valued inner product and that it is a left $\mathscr{D}_1$, right $\mathscr{D}_H$, bimodule which fits Rieffel's framework, and thus provides the basis for our construction of induced representations. In Section 5 we identify the algebra $K(X)$ as the crossed product $(A \times_\delta G) \times_{\hat{\delta}} H$, where $\hat{\delta}$ is the dual action of G on $A \times_\delta G$ (restricted to H). This enables us to establish the following imprimitivity theorem for the induction process: A representation v of $A \times_\delta G$ on $\mathscr{H}$ is induced from a representation of $A \times_{\delta|} (G/H)$ if, and only if, there exists a unitary representation U of H on $\mathscr{H}$ such that (v, U) is a covariant representation of $(A \times_\delta G, H, \hat{\delta})$. In the remaining two sections we investigate the continuity of our induction process and show that the dual action to a coaction is proper and saturated, confirming Rieffel's conjecture.

This paper forms part of the author's doctoral thesis, which was submitted to the University of New South Wales in September 1988. The author takes this opportunity to thank his graduate adviser, Professor Iain Raeburn.

1. Preliminaries

Firstly we establish some notation. Throughout, A and B will be C^*-algebras, $M(A)$ will denote the multiplier algebra of A, and $B(\mathscr{H})$ will be the algebra of linear operators on the Hilbert space $\mathscr{H}$, with $K(\mathscr{H})$ the ideal of compact operators. G will be a locally compact group, with λ_G and ρ_G, respectively, the left and right regular representations of G, and $L^1(G)$, in $B(L^2(G))$. Unless otherwise stated, all representations will be assumed non-degenerate. $C^*(G)$ will denote the group C^*-algebra and $C_r^*(G)$ will be the subalgebra $\lambda_G(C^*(G))$ of $B(L^2(G))$. The natural inclusion of G in $M(C^*(G))$ will be denoted by i_G. We will denote the complex and real numbers by $\mathbb{C}$ and $\mathbb{R}$, respectively. $C_b(G, A)$, $C_0(G, A)$, and $C_c(G, A)$ will denote the continuous functions from G to A which (i) are bounded, (ii) vanish at infinity, and (iii) have compact support. If A is $\mathbb{C}$, then the above algebras will be denoted $C_b(G)$, $C_0(G)$, and $C_c(G)$, respectively. $C_b^s(G, A)$ will denote the bounded strictly continuous maps from G to $M(A)$. M_G will denote the representation of $C_b(G)$ on $L^2(G)$ by multiplication operators. If B is a Banach space, B^* will denote the dual space of B. If f is a function on G, we define $\bar{f}$, $\check{f}$, $\tilde{f}$, f_s, and f^s by $\bar{f}(t)=\overline{f(t)}$, $\check{f}(t)=f(t^{-1})$, $\tilde{f}(t)=\overline{f(t^{-1})}$, $f_s(t)=f(s^{-1}t)$, and $f^s(t)=f(ts)$.

Let ds denote left Haar measure on G. A map $f\colon G\to B(\mathscr{H})$ is (weakly) integrable if the maps: $s\to\omega(f(s))$ are Lebesgue integrable and there exists an element $T\in B(\mathscr{H})$ such that $\omega(T)=\int_G \omega(f(s))\,ds$ for all weakly continuous functionals ω on $B(\mathscr{H})$. The element T is unique and will be denoted $\int_G f(s)\,ds$. All integrals of operator valued functions will be of this type. We note that all weakly continuous compactly supported maps f: $G\to B(\mathscr{H})$ are integrable.

Slice Maps

Let $A\odot B$ denote the algebraic tensor product of A and B. Let $A\otimes B$ denote its completion in the minimum C^*-norm. For each $u\in B^*$ we define the slice map $S_u\colon A\odot B\to A$ by $\sum_{i=1}^n a_i\otimes b_i\to\sum_{i=1}^n u(b_i)a_i$. By [27, Theorem 1], S_u is bounded for the minimum C^*-norm and hence extends to $A\otimes B$. Define left and right actions of B on B^* by $(b\bullet u)(a)=u(ab)$ and $(u\bullet b)(a)=u(ba)$, for a, $b\in B$. It is easily checked that

$$S_{u\bullet b}(z)=S_u((1\otimes b)z) \qquad S_{b\bullet u}(z)=S_u(z(1\otimes b))$$
$$aS_u(z)=S_u((a\otimes 1)z) \qquad S_u(z)a=S_u(z(a\otimes 1)),$$

where $z\in A\otimes B$. Now B^* is a two sided B-module with a bounded approximate identity such that $\|b\bullet u\|_{B^*}\leqslant\|b\|_B\ \|u\|_{B^*}$. So by Cohen's factorisation theorem [10, Theorem 32.22] any element u of B^* can be written as $u=b\bullet\psi=c\bullet v\bullet d=w\bullet e$, for some v, w, $\psi\in B^*$ and b, c, d, $e\in B$. We can use this to extend S_u to a map $\mathscr{S}_u\colon M(A\otimes B)\to M(A)$ determined by $\mathscr{S}_u(z)a = \mathscr{S}_{b\bullet v\bullet c}(z)a = \mathscr{S}_{v\bullet c}(z(a\otimes b))$ and $a\mathscr{S}_u(z) = a\mathscr{S}_{b\bullet v\bullet c}(z) = \mathscr{S}_{b\bullet v}((a\otimes c)z)$. It is easily checked that the definition is independent of the way we factor u, that $\mathscr{S}_u$ is strictly continuous, that $\|\mathscr{S}_u\| = \|u\|_{B^*}$, and that the formulas (1) still hold [14, Lemma 1.5]. Define a map $\gamma\colon C^*(G)^*\to C_b(G)$ by $\{\gamma(\psi)\}(s)=\psi(i_G(s))$. Let $B(G)=\gamma(C^*(G)^*)$, $B_r(G)=\gamma(C_r^*(G)^*)$, and $A(G)=\gamma(vN(G)_*)$, where $vN(G)_*$ is the predual of the group von Neumann algebra $vN(G)$. $B(G)$ equipped with the algebraic operations of $C_b(G)$ and the norm inherited from $C_r^*(G)^*$ is an involutive Banach algebra with subalgebras $B_r(G)$ and $A(G)$, see [4, 19, Sect. 7.1]. $A(G)$ is called the Fourier algebra. We will denote the compactly supported elements of $A(G)$ by $A_c(G)$. We shall mainly be interested in slice maps when B is $C_r^*(G)$ and B^* is $B_r(G)$. Note that if $u \in A(G) \subset B_r(G)$, $s \in G$, and $g \in C_c(G)$, then $u(\lambda_G(s)) = u(s)$ and $u(\lambda_G(g)) = \int_G u(s)\, g(s)\, ds$.

Induced Representations of C^*-Algebras

Let $\mathscr{A}$ and $\mathscr{B}$ be pre-C^*-algebras with completions A and B, respectively. Let $\mathscr{D}$ be a $\mathscr{B}$-rigged space with $\mathscr{B}$-action $\bullet$ and pre-$\mathscr{B}$-valued inner product $\langle \cdot, \cdot \rangle_{\mathscr{D}}$. We will denote the C^*-algebra of (equivalence classes of) bounded operators on $\mathscr{D}$ by $L(\mathscr{D})$ and the imprimitivity algebra of $\mathscr{D}$, that is, the closure in $L(\mathscr{D})$ of the operators $T_{w,x}$: $\mathscr{D} \to \mathscr{D}$: $y \to w \bullet \langle x, y \rangle_{\mathscr{D}}$, where w, $x \in \mathscr{D}$, by $K(\mathscr{D})$. We note that $K(\mathscr{D})$ is strongly Morita equivalent to B [21, Propositions 6.5 and 6.6]. Now suppose $\mathscr{D}$ is a pre-Hermitian $\mathscr{B}$-rigged $\mathscr{A}$-module with $\mathscr{A}$-action $\bullet$. Then $\mathscr{D}$ can be used to construct representations of A from representations of B. This is achieved as follows: Suppose v is a representation of B on $\mathscr{H}$. Then we define a pre-inner product on $\mathscr{D} \otimes \mathscr{H}$ by $\langle x \otimes \xi, y \otimes \eta \rangle_{\mathscr{D} \otimes \mathscr{H}} = \langle \{v(\langle y, x \rangle_{\mathscr{D}})\}(\xi), \eta \rangle_{\mathscr{H}}$, for ξ, $\eta \in \mathscr{H}$ (this is equivalent to Rieffel's construction [21, Theorem 5.1], which uses $\mathscr{D} \otimes_{\mathscr{B}} \mathscr{H}$ instead of $\mathscr{D} \otimes \mathscr{H}$, since the elements $(x \bullet b) \otimes \xi - x \otimes (b \bullet \xi)$ have length zero). We obtain a Hilbert space $\mathscr{D}\text{-ind}_B^A \mathscr{H}$ from $\mathscr{D} \otimes \mathscr{H}$ by factoring out the vectors of length zero and completing. The representation $\mathscr{D}\text{-ind}_B^A v$ of A on $\mathscr{D}\text{-ind}_B^A \mathscr{H}$ determined by $\{\{\mathscr{D}\text{-ind}_B^A v\}(a)\}(x \otimes \xi) = (a \bullet x) \otimes \xi$ is called the representation induced from v. For any $\mathscr{B}$-rigged space $\mathscr{D}$, $\|x\|_{\mathscr{D}}^2 = \|\langle x, x \rangle_{\mathscr{D}}\|_B$ defines a semi-norm on $\mathscr{D}$. Factoring by the vectors of length zero and completing we obtain a Banach space X. It can be checked that X is a B-rigged space and that $K(X)$ is isomorphic to $K(\mathscr{D})$. Further if $\mathscr{D}$ is a pre-Hermitian $\mathscr{B}$-rigged $\mathscr{A}$-module, then X is a pre-Hermitian B-rigged A-module and can thus also be used to induce representations of B to representations of A. It is not difficult to check that the representations $\mathscr{D}\text{-ind}_B^A v$ and $X\text{-ind}_B^A v$ are unitarily equivalent. When $\mathscr{D}$ (or X) is understood we will denote the induced representation simply by $\text{ind}_B^A v$. Although it is clearly easier to construct induced representations using the module $\mathscr{D}$, rather than its completion X, we shall use X since it will be necessary to do so when we study the continuity of the induction process. For more on the above, see [21, 22, Sect. 3].

Coactions and Their Crossed Products

We will call a homomorphism γ: $A \to M(B)$ non-degenerate if A has an approximate identity $(e_i)_{i \in I}$ such that $\gamma(e_i) \to 1$ strictly in $M(B)$. If γ is non-degenerate, then it has a unique strictly continuous extention, which we will continue to denote γ, to $M(A)$ [14, Lemma 1.1]. $\tilde{M}(A \otimes B)$ will denote the set of $x \in M(A \otimes B)$ such that $x(1 \otimes z)$ and $(1 \otimes z)x \in A \otimes B$ for all

$z \in B$. Let $g \in C_c(G)$. Then δ_G: $C_r^*(G) \to \tilde{M}(C_r^*(G) \otimes C_r^*(G))$: $\lambda_G(g) \to \int_G g(s)\, \lambda_G(s) \otimes \lambda_G(s)\, ds$ determines a well-defined $*$-homomorphism called the comultiplication map. A coaction of G on A is an injective non-degenerate $*$-homomorphism δ: $A \to \tilde{M}(A \otimes C_r^*(G))$ such that $(\delta \otimes i) \circ \delta = (i \otimes \delta_G) \circ \delta$. We will call a coaction non-degenerate if for each $\zeta \in A^*$ there exists $\psi \in C_r^*(G)^*$ such that $(\zeta \otimes \psi) \circ \delta \neq 0$. If π is a faithful representation of A on $\mathscr{H}$ and δ is a coaction of G on A, then the crossed product $A \times_\delta G$ of A, by δ, is the C^*-subalgebra of $B(\mathscr{H} \otimes L^2(G))$ generated by the elements $\{\pi \otimes i\}(\delta(a))(1 \otimes M_G(f))$, where $a \in A$ and $f \in C_0(G)$. It can be shown that the crossed product is independent of the choice of π. When there is no danger of confusion $\{\pi \otimes i\}(\delta(a))(1 \otimes M_G(f))$ will be shortened to $\delta(a)(1 \otimes f)$ and $(1 \otimes M_G(f))\{\pi \otimes i\}(\delta(a))$ to $(1 \otimes f)\, \delta(a)$. For any coaction δ there is a natural action, the dual action, $\hat{\delta}$ of G on $A \times_\delta G$ defined by $\hat{\delta}_s = Ad(1 \otimes \rho_G(s))$. We define a unitary element ω_G of $C_b^s(G, C_r^*(G))$ $(\cong M(C_0(G) \otimes C_r^*(G)))$ by $\omega_G(s) = \lambda_G(s)$, for all $s \in G$. A covariant representation for the system (A, G, δ) on $\mathscr{H}$ is a pair (π, μ), where π and μ are representations of A and $C_0(G)$, respectively, on $\mathscr{H}$, such that

$$\{\pi \otimes i\}(\delta(a)) = (\{\mu \otimes i\}(\omega_G))(\pi(a) \otimes 1)(\{\mu \otimes i\}(\omega_G^*)) \qquad \forall a \in A,$$

see [20]. We note that $\mathscr{S}_u(\omega_G) = u$ and hence that $\mathscr{S}_u(\{\mu \otimes i\}(\omega_G)) = \mu(u)$. To see that the covariance condition of [14, Definition 3.5] is the same as that presented here, note that the corepresentation $W \in B(\mathscr{H}) \bar{\otimes} vN(G)$ corresponding to the representation μ is determined by the relationship $\mathscr{S}_u(W) = \mu(u)$, for all $u \in A(G)$, so that $W = \{\mu \otimes i\}(\omega_G)$. In [14] it is proved that the representations of $A \times_\delta G$ on $\mathscr{H}$ correspond bijectively to the covariant representations of (A, G, δ) on $\mathscr{H}$. We denote the representation of $A \times_\delta G$ corresponding to the covariant representation (π, μ) of (A, G, δ) by $\pi \times \mu$. By [14, Theorem 3.7] we have that $(\pi \times \mu) \circ \delta = \pi$ and $(\pi \times \mu) \circ (1 \otimes M_G) = \mu$.

Let $u \in B_r(G)$. If $\mathscr{S}_u$: $M(A \otimes C_r^*(G)) \to M(A)$ is a slice map, then we will abbreviate $\mathscr{S}_u(\delta(a))$ by $\delta_u(a)$. Note that $\mathscr{S}_u(\tilde{M}(A \otimes C_r^*(G))) \subset A$.

LEMMA 1. *Suppose* a, $b \in A$, u, v, $w \in A_c(G)$, *that* w *is one on* $(\operatorname{supp} u)(\operatorname{supp} v)$, *and that*

$$\delta(a) = \operatorname*{strict\ limit} \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(f_{ij}) \qquad \text{and} \qquad \delta(b) = \operatorname*{strict\ limit} \sum_{l=1}^{m_k} b_{kl} \otimes \lambda_G(g_{kl})$$

in $M(A \otimes C_r^*(G))$ *for some* a_{ij}, $b_{kl} \in A$ *and* f_{ij}, $g_{kl} \in C_c(G)$. *Then if* $\cdot$ *denotes pointwise multiplication*

(i) $\delta(\delta_v(a))$ *is the strict limit of the* $\sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(v \cdot f_{ij})$,

(ii) $\delta_u(\delta_v(a)) = \delta_{u \cdot v}(a)$,

(iii) $\delta_w(\delta_u(a)\,\delta_v(b)) = \delta_u(a)\,\delta_v(b)$,

(iv) $(\mathcal{S}_u(z))^* = \mathcal{S}_{\bar{u}}(z^*)$ *for all* $z \in M(A \otimes C_r^*(G))$.

Proof. (i)

$$\begin{aligned}
\delta(\delta_v(a)) &= \{i \otimes \mathcal{S}_v\}(\{\delta \otimes i\}(\delta(a))) \\
&= \{i \otimes \mathcal{S}_v\}(\{i \otimes \delta_G\}(\delta(a))) \\
&= \lim_{i \in I} \sum_{j=1}^{n_i} a_{ij} \otimes \mathcal{S}_v\left(\int_G f_{ij}(s)\,\lambda_G(s) \otimes \lambda_G(s)\,ds\right) \\
&= \lim_{i \in I} \sum_{j=1}^{n_i} a_{ij} \otimes \left(\int_G v(s)\,f_{ij}(s)\,\lambda_G(s)\,ds\right) \\
&= \lim_{i \in I} \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(v \cdot f_{ij}).
\end{aligned}$$

(ii)
$$\begin{aligned}
\delta_u(\delta_v(a)) &= \lim_{i \in I} \sum_{j=1}^{n_i} a_{ij} u(\lambda_G(v \cdot f_{ij})) \\
&= \lim_{i \in I} \sum_{j=1}^{n_i} a_{ij}((u \cdot v)(\lambda_G(f_{ij}))) = \delta_{u \cdot v}(a).
\end{aligned}$$

(iii) By Cohen's factorization theorem we can write $w = c \bullet \psi \bullet d$ for some c, $d \in C_r^*(G)$ and $\psi \in B_r(G)$. Let $e \in A$. Then

$$\begin{aligned}
e\,\delta_w(\delta_u(a)\,\delta_v(b))e &= e\,\delta_{c \bullet \psi \bullet d}(\delta_u(a)\,\delta_v(b))e \\
&= \mathcal{S}_\psi((e \otimes d)\,\delta(\delta_u(a))\,\delta(\delta_v(b))(e \otimes c)) \\
&= \mathcal{S}_\psi\left(\left(\lim_{i \to \infty} \sum_{j=1}^{n_i} (ea_{ij}) \otimes (d\lambda_G(u \cdot f_{ij}))\right)\right. \\
&\qquad \left.\times\left(\lim_{k \to \infty} \sum_{l=1}^{m_k} (b_{kl}e) \otimes (\lambda_G(v \cdot g_{kl})c)\right)\right) \\
&= \lim_{i \to \infty} \sum_{j,l} ea_{ij}b_{il}e\ w(\lambda_G(u \cdot f_{ij}) * \lambda_G(v \cdot g_{il})) \\
&= \lim_{i \to \infty} \sum_{j,l} ea_{ij}b_{il}e\ u(\lambda_G(f_{ij}))\,v(\lambda_G(g_{il}))
\end{aligned}$$

(by a straightforward calculation using the fact that w is one on the set $(\operatorname{supp} u)(\operatorname{supp} v)$)

$$= e\,\delta_u(a)\,\delta_v(b)e.$$

Now letting e run over an approximate identity of A gives (iii).

(iv) Suppose $z=\sum_{i=1}^{n} a_i \otimes \lambda_G(g_i) \in A \odot \lambda_G(C_c(G))$. Then

$$(\mathscr{S}_u(z))^* = \sum_{i=1}^{n} a_i^* \overline{u(\lambda_G(g_i))} = \sum_{i=1}^{n} a_i^* \tilde{u}(\lambda_G(g_i^*))$$
$$= \mathscr{S}_{\tilde{u}}\left(\left(\sum_{i=1}^{n} a_i \otimes \lambda_G(g_i)\right)^*\right) = \mathscr{S}_{\tilde{u}}(z^*)$$

and (iv) follows since $\mathscr{S}_u$, $\mathscr{S}_{\tilde{u}}$, and the $*$-operation are strictly continuous. ∎

2. Coactions of Quotients

Throughout, δ will be a coaction of G on A and π will be a faithful representation of A on the Hilbert space $\mathscr{H}$. In this section H will be a closed normal amenable subgroup of G. For each such H we define a coaction $\delta|: A \to \tilde{M}(A \otimes C_r^*(G/H))$ of G/H on A. We then show that the crossed product $A \times_{\delta|} (G/H)$ has a faithful representation on $B(\mathscr{H} \otimes L^2(G))$.

Define φ: $C_c(G) \to C_c(G/H)$ by $\{\varphi(f)\}(sH) = \int_H f(sh)\,dh$. Note that φ is surjective [9, Theorem 15.21]. We will normalize the Haar measures on G and G/H by insisting that

$$\int_G f(s)\,d\mu_G(s) = \int_{G/H} \int_H f(sh)\,d\mu_H(h)\,d\mu_{G/H}(sH) \qquad \forall f \in C_c(G).$$

In order to define $\delta|$ we need to prove the following two lemmas. Let $\operatorname{ind}_{C^*(H)}^{C^*(G)} \nu$ denote the representation of $C^*(G)$ induced from the representation ν of $C^*(H)$ as in [21, Sect. 7].

Lemma 2. *Suppose 1_H is the integrated form of the trivial representation of H on $\mathbb{C}$ and Λ: $C^*(G) \to M(C_r^*(G/H))$ is the integrated form of the unitary representation $\wp$: $G \to M(C_r^*(G/H))$: $s \to \lambda_{G/H}(sH)$. Then Λ is unitarily equivalent to $\operatorname{ind}_{C^*(H)}^{C^*(G)} 1_H$.*

Proof. Let f, x, $y \in C_c(G)$ and α, $\beta \in \mathbb{C}$. Recall that $\operatorname{ind}_{C^*(H)}^{C^*(G)} 1_H$ is a representation of $C^*(G)$ on Z, where Z is $C_c(G) \otimes \mathbb{C}$ factored and completed with respect to the pre-inner product $\langle x \otimes \alpha, y \otimes \beta \rangle_{C_c(G) \otimes \mathbb{C}} = \langle \{1_H(\langle y, x \rangle_{C_c(G)})\}(\alpha), \beta \rangle_{\mathbb{C}}$. Define U: $C_c(G) \otimes \mathbb{C} \to L^2(G/H)$ by $x \otimes 1 \to \varphi(x)$. Since $\varphi(C_c(G)) = C_c(G/H)$, U maps onto a dense subspace of $L^2(G/H)$. Now

$$\begin{aligned}\langle x\otimes 1, y\otimes 1\rangle_{C_c(G)\otimes\mathbb{C}} &= \int_H \langle y, x\rangle_{C_c(G)}(h)\,dh\\ &= \int_H \int_G \overline{y(s)}\,x(sh)\,ds\,dh\\ &= \int_{G/H}\left(\int_H \overline{y(sr)}\,dr\right)\left(\int_H x(sh)\,dh\right)dsH\\ &= \langle U(x\otimes 1), U(y\otimes 1)\rangle_{L^2(G/H)}.\end{aligned}$$

So U preserves the pre-inner products. We also have that

$$\begin{aligned}\{U(\{\operatorname{ind}_{C^*(H)}^{C^*(G)} 1_H(f)\}(x\otimes 1))\}(tH) &= \{U((f\bullet x)\otimes 1)\}(tH)\\ &= \int_H \int_G f(s)\,x(s^{-1}th)\,ds\,dh\\ &= \left\{\left\{\int_G f(s)\,\lambda_{G/H}(sH)\,ds\right\}(\varphi(x))\right\}(tH)\\ &= \{\{\Lambda(f)\}(U(x\otimes 1))\}(tH).\end{aligned}$$

Hence U extends to a unitary operator from Z onto $L^2(G/H)$ which intertwines the $C^*(G)$ actions and the two representations are unitarily equivalent as claimed. ∎

LEMMA 3. *Suppose* Λ: $C^*(G)\to C_r^*(G/H)$ *is as in Lemma* 2. *Then* $\ker\Lambda\supset\ker\lambda_G$. *Hence the map* Φ: $C_r^*(G)\to C_r^*(G/H)$ *determined by* $\lambda_G(s)\to\lambda_{G/H}(sH)$ *is well defined and* $\Lambda=\Phi\circ\lambda_G$. *Also* Φ *is a non-degenerate* $*$-*homomorphism such that* $\Phi(\lambda_G(f))=\lambda_{G/H}(\varphi(f))$ *for all* $f\in C_c(G)$.

Proof. Let λ_H be the left regular representation of $C^*(H)$ on $L^2(H)$. Then since H is amenable $\ker\lambda_H$ is trivial [19, Theorem 7.7.5]. Hence $\ker 1_H$ contains $\ker\lambda_H$. By Lemma 2 and since induction preserves weak containment [8, Proposition 9], we have that

$$\ker\Lambda=\ker(\operatorname{ind}_{C^*(H)}^{C^*(G)} 1_H)\supset\ker(\operatorname{ind}_{C^*(H)}^{C^*(G)}\lambda_H)=\ker\lambda_G.$$

The last statement of the lemma is easily checked. It implies that $\Phi(C_r^*(G))$ is contained in the closure of $\lambda_G(C_c(G/H))$, which is $C_r^*(G/H)$. That is, Φ maps into $C_r^*(G/H)$. To see that Φ is a non-degenerate $*$-homomorphism, let $(e_j)_{j\in J}$ be an approximate identity for $C_r^*(G)$ contained in $C_c(G)$, and let $g=\varphi(f)\in C_c(G/H)$ for some $f\in C_c(G)$. Then

$$\Phi(\lambda_G(e_j))\,\lambda_{G/H}(g)=\Phi(\lambda_G(e_j*f))\to\Phi(\lambda_G(f))=\lambda_{G/H}(g).$$

Similarly $\lambda_{G/H}(g)\,\Phi(\lambda_G(e_j))\to\lambda_{G/H}(g)$. Hence, since such $\lambda_{G/H}(g)$ are dense, $\Phi(\lambda_G(e_j))\to 1$ strictly in $M(C_r^*(G/H))$ and Φ is non-degenerate as claimed. ∎

LEMMA 4. *Suppose δ is a coaction of G on A. Define*

$$\delta| : A \to \tilde{M}(A \otimes C_r^*(G/H)) \qquad by \quad \delta|\,(a) = \{i \otimes \Phi\}(\delta(a)).$$

Then $\delta|$ is a coaction of G/H on A.

Proof. Since Φ is non-degenerate, so is $i \otimes \Phi$. Hence $i \otimes \Phi$ extends to $M(A \otimes C_r^*(G))$. Since both δ and Φ are non-degenerate $*$-homomorphisms, so is $\delta|$. Let $f \in C_c(G)$. To see that $\delta|$ maps into $\tilde{M}(A \otimes C_r^*(G/H))$, note that by Lemma 3

$$\begin{aligned}\delta|\,(a)(1 \otimes \lambda_{G/H}(\varphi f)) &= \{i \otimes \Phi\}(\delta(a))\{i \otimes \Phi\}(1 \otimes \lambda_G(f)) \\ &= \{i \otimes \Phi\}(\delta(a)(1 \otimes \lambda_G(f))) \in A \otimes C_r^*(G/H),\end{aligned}$$

since $\delta(a)(1 \otimes \lambda_G(f)) \in A \otimes C_r^*(G)$. Hence $\delta|\,(a)(1 \otimes z) \in A \otimes C_r^*(G/H)$ for all $z \in C_r^*(G/H)$, since elements of the form $\lambda_{G/H}(\varphi f)$ are dense in $C_r^*(G/H)$. Now

$$\begin{aligned}\delta_G|\,(\lambda_G(f)) &= \{i \otimes \Phi\}(\delta_G(\lambda_G(f))) \\ &= \{i \otimes \Phi\}\left(\int_G f(s)\,\lambda_G(s) \otimes \lambda_G(s)\,ds\right) \\ &= \int_G f(s)\,\lambda_G(s) \otimes \lambda_{G/H}(sH)\,ds \\ &= W(\lambda_G(f) \otimes 1)\,W^*,\end{aligned}$$

where W is the unitary operator in $B(L^2(G \times G/H))$ defined by $\{W(\xi)\}(s, tH) = \xi(s, s^{-1}tH)$. The right hand side of the above is clearly injective, hence so is $\delta_G|$. Now since

$$\begin{aligned}(i \otimes \delta_G|) \circ \delta &= (i \otimes i \otimes \Phi) \circ (i \otimes \delta_G) \circ \delta \\ &= (i \otimes i \otimes \Phi) \circ (\delta \otimes i) \circ \delta \\ &= (\delta \otimes i) \circ \delta|,\end{aligned}$$

the injectivity of $(i \otimes \delta_G|) \circ \delta$ implies that of $(\delta \otimes i) \circ \delta|$, and hence of $\delta|$. It remains to show the coaction identity. Firstly, we note that

$$\begin{aligned}(\Phi \otimes \Phi) \circ \delta_G(\lambda_G(f)) &= \{\Phi \otimes \Phi\}\left(\int_G f(s)\,\lambda_G(s) \otimes \lambda_G(s)\,ds\right) \\ &= \int_G f(s)\,\lambda_{G/H}(sH) \otimes \lambda_{G/H}(sH)\,ds \\ &= \int_{G/H} \int_H f(sh)\,dh\,\lambda_{G/H}(sH) \otimes \lambda_{G/H}(sH)\,dsH \\ &= \delta_{G/H} \circ \Phi(\lambda_G(f)).\end{aligned}$$

Hence

$$\begin{aligned}(\delta| \otimes i)\circ\delta| &= (i\otimes\Phi\otimes\Phi)\circ(\delta\otimes i)\circ\delta\\ &= (i\otimes\Phi\otimes\Phi)\circ(i\otimes\delta_G)\circ\delta\\ &= (i\otimes\delta_{G/H})\circ(i\otimes\Phi)\circ\delta\\ &= (i\otimes\delta_{G/H})\circ\delta|. \quad \blacksquare\end{aligned}$$

It should be noted that if δ is a coaction of an abelian group G and α is the corresponding action of $\hat{G}$, then $\delta|$ corresponds to the restriction of α to the subgroup $H^{\perp}$ (explaining the notation) and that $A\times_{\delta|}(G/H)$ is isomorphic to $A\times_{\alpha}H^{\perp}$. In what follows we shall embed $C_0(G/H)$ in $C_b(G)$ via the mapping q: $C_0(G/H)\to C_b(G)$, where $\{q(F)\}(s)=F(sH)$, for $s\in G$, and $F\in C_0(G)$. That is, we shall view elements of $C_0(G/H)$ as functions in $C_b(G)$ which are constant on H-cosets.

PROPOSITION 5. *Suppose (ϱ, μ) is a covariant representation of (A, G, δ) on a Hilbert space $\mathcal{Q}$. Then $(\varrho, \mu\circ q)$ is a covariant representation of $(A, G/H, \delta|)$ on $\mathcal{Q}$. Hence we have a map, which we will call the restriction map,*

$$\mathrm{Res}^{A\times_\delta G}_{A\times_{\delta|}(G/H)}\colon \mathrm{Rep}(A\times_\delta G)\to \mathrm{Rep}(A\times_{\delta|}(G/H))\colon \varrho\times\mu\to\varrho\times(\mu\circ q).$$

Proof. Firstly, we note that since μ is non-degenerate it has a (unique strictly continuous) extension to $M(C_0(G))\cong C_b(G)$. So $\mu\circ q$ is well defined. Now

$$\{\{i\otimes\Phi\}(\omega_G)\}(s)=\Phi(\omega_G(s))=\omega_{G/H}(sH)=\{\{q\otimes i\}(\omega_{G/H})\}(s).$$

This and the fact that (ϱ, μ) is a covariant representation gives

$$\begin{aligned}\{\varrho\otimes i\}(\delta|(a)) &= \{i\otimes\Phi\}(\{\varrho\otimes i\}(\delta(a)))\\ &= \{i\otimes\Phi\}((\{\mu\otimes i\}(\omega_G))(\varrho(a)\otimes 1)(\{\mu\otimes i\}(\omega_G^*)))\\ &= (\{(\mu\circ q)\otimes i\}(\omega_{G/H}))(\varrho(a)\otimes 1)(\{(\mu\circ q)\otimes i\}(\omega_{G/H}^*)),\end{aligned}$$

thus $(\varrho, \mu\circ q)$ is a covariant representation of $(A, G/H, \delta|)$. $\blacksquare$

LEMMA 6. *Suppose δ is a coaction of G on A and ϱ is a representation of A on $\mathcal{Q}$. Then $((\varrho\otimes i)\circ\delta, 1\otimes M_G)$ is a covariant representation of (A, G, δ) on $\mathcal{Q}\otimes L^2(G)$.*

Proof. Firstly we show that if i is the identity map on $C_r^*(G)$, then

(i, M_G) is a covariant representation of $(C_r^*(G), G, \delta_G)$ on $L^2(G)$. Let $u \in A(G)$. By (1) and the fact that $\mathscr{S}_u(\{\mu \otimes i\}(\omega_G)) = \mu(u)$, we have

$$\begin{aligned}\mathscr{S}_u(\delta_G(\lambda_G(s))\{M_G \otimes i\}(\omega_G)) &= \mathscr{S}_u((\lambda_G(s) \otimes \lambda_G(s))\{M_G \otimes i\}(\omega_G)) \\ &= \lambda_G(s)\, \mathscr{S}_{u \bullet \lambda_G(s)}(\{M_G \otimes i\}(\omega_G)) \\ &= \lambda_G(s)\, M_G(u \bullet \lambda_G(s)) \\ &= \lambda_G(s)\, M_G(\tau_s^{-1}(u)),\end{aligned}$$

where τ is the left translation action of G on $C_0(G)$. Also

$$\begin{aligned}\mathscr{S}_u(\{M_G \otimes i\}(\omega_G)(\lambda_G(s) \otimes 1)) &= \mathscr{S}_u(\{M_G \otimes i\}(\omega_G))\, \lambda_G(s) \\ &= M_G(u)\, \lambda_G(s).\end{aligned}$$

Now by the above and since (M_G, λ_G) is a covariant representation of the system $(C_0(G), G, \tau)$ we have that

$$\mathscr{S}_u(\delta_G(\lambda_G(s))\{M_G \otimes i\}(\omega_G)) = \mathscr{S}_u(\{M_G \otimes i\}(\omega_G)(\lambda_G(s) \otimes 1)),$$

and since u was arbitrary in $A(G)$, that

$$\delta_G(\lambda_G(s))\{M_G \otimes i\}(\omega_G) = \{M_G \otimes i\}(\omega_G)(\lambda_G(s) \otimes 1).$$

Integrating we see that (i, M_G) is a covariant representation as claimed. This, and the coaction identity allows us to deduce the covariance of the pair $((\varrho \otimes i) \circ \delta, 1 \otimes M_G)$ as

$$\begin{aligned}&\{((\varrho \otimes i) \circ \delta) \otimes i\}(\delta(a)) \\ &\quad = \{\varrho \otimes i \otimes i\}(\{i \otimes \delta_G\}(\delta(a))) \\ &\quad = (1 \otimes \{M_G \otimes i\}(\omega_G))(\{\varrho \otimes i\}(\delta(a)) \otimes 1)(1 \otimes \{M_G \otimes i\}(\omega_G^*)) \\ &\quad = \{1 \otimes M_G \otimes i\}(\omega_G)(\{\varrho \otimes i\}(\delta(a)) \otimes 1)\{1 \otimes M_G \otimes i\}(\omega_G^*). \quad \blacksquare\end{aligned}$$

In [21, Sect. 7] Rieffel shows that $C_c(G)$ is an equivalence bimodule implementing a strong Morita equivalence between $C_0(G/H) \times_\tau G$ and $C^*(H)$. It is readily seen that $(M_G \circ q) \times \lambda_G$ is the representation of $C_0(G/H) \times_\tau G$ induced from λ_H. Hence by [8, Proposition 9] and the fact that H is amenable

$$\begin{aligned}\ker((M_G \circ q) \times \lambda_G) &= \ker(\operatorname{ind}_{C^*(H)}^{C_0(G/H) \times_\tau G} \lambda_H) \\ &= \operatorname{ind}_{C^*(H)}^{C_0(G/H) \times_\tau G}(\ker \lambda_H) = 0,\end{aligned}$$

which shows that the representation $(M_G \circ q) \times \lambda_G$: $C_0(G/H) \times_\tau G \rightarrow$

$B(L^2(G))$ is faithful. We will need this fact in proving the following proposition.

PROPOSITION 7. *Suppose δ is a coaction of G on A and H is a closed normal amenable subgroup of G. Then the map*

$$\Gamma = \mathrm{Res}^{A \times_\delta G}_{A \times_{\delta|}(G/H))} (((\pi \otimes i) \circ \delta) \times (1 \otimes M_G))$$
$$: A \times_{\delta|} (G/H) \to B(\mathscr{H} \otimes L^2(G))$$

is a faithful representation of $A \times_{\delta|} (G/H)$ on $B(\mathscr{H} \otimes L^2(G))$.

Proof. Lemma 6 shows that $((\pi \otimes i) \circ \delta,\ 1 \otimes M_G)$ is a covariant representation of (A, G, δ) on $B(\mathscr{H} \otimes L^2(G))$. So, by Proposition 5, Γ is a covariant representation of $A \times_{\delta|} (G/H)$. Since H is amenable, the representation

$$\gamma = \pi \otimes ((M_G \circ q) \times \lambda_G) : M(A \otimes (C_0(G/H) \times_\tau G)) \to B(\mathscr{H} \otimes L^2(G))$$

is injective and thus has an inverse. We will show that

$$(\pi \otimes (M_{G/H} \times \wp)) \circ (\pi \otimes ((M_G \circ q) \times \lambda_G))^{-1} \circ \Gamma = \mathrm{id}_{A \times_{\delta|}(G/H)},$$

where $\wp$ is as in Lemma 2, which will imply that Γ is injective, as required. Suppose $\{\pi \otimes i\}(\delta| \,(a))(1 \otimes M_{G/H}(F))$ is a generator of $A \times_{\delta|} (G/H)$ and that $\delta(a)$ is the strict limit of the net $(\sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}))$, where the $a_{ij} \in A$ and $g_{ij} \in C_c(G)$. Then

$$\begin{aligned}
&(\pi \otimes (M_{G/H} \times \wp)) \circ \gamma^{-1} \circ \Gamma(\{\pi \otimes i\}(\delta| \,(a))(1 \otimes M_{G/H}(F))) \\
&= (\pi \otimes (M_{G/H} \times \wp)) \\
&\quad \circ \gamma^{-1}(\{\pi \otimes i\}(\delta(a))(1 \otimes (M_G \circ q(F)))) \\
&= (\pi \otimes (M_{G/H} \times \wp)) \\
&\quad \circ \gamma^{-1} \left(\begin{matrix}\text{strong} \\ \text{limit}\end{matrix} \sum_{j=1}^{n_i} \pi(a_{ij}) \otimes (\lambda_G(g_{ij})(M_G \circ q(F))) \right) \\
&= \{\pi \otimes (M_{G/H} \times \wp)\} \left(\begin{matrix}\text{strict} \\ \text{limit}\end{matrix} \sum_{j=1}^{n_i} a_{ij} \otimes (i_G(g_{ij})\, i_{C_0(G/H)}(F)) \right)
\end{aligned}$$

(i_G and $i_{C_0(G/H)}$ are the usual embeddings of G and $C_0(G/H)$ in $M(C_0(G/H) \times_\tau G))$

$$= \{\pi \otimes \Phi\} \left(\begin{matrix}\text{strict} \\ \text{limit}\end{matrix} \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}) \right) (1 \otimes M_{G/H}(F))$$

(since the integrated form, Λ, of $\wp$ equals $\Phi \circ \lambda_G$)

$$= \{\pi \otimes i\}(\delta| \, (a))(1 \otimes M_{G/H}(F)),$$

and the proof is complete. ▮

Henceforth we will suppress the map q, in particular, $M_G(q(F))$ will be denoted $M_G(F)$. The image $\Gamma(A \times_{\delta|} (G/H)) \subset B(\mathscr{H} \otimes L^2(G))$ of $A \times_{\delta|} (G/H)$ under Γ will be denoted by $A \times_\delta (G/H)$. From Proposition 7 it is clear that $A \times_\delta (G/H)$ is the C^*-subalgebra of $B(\mathscr{H} \otimes L^2(G))$ generated by the elements $\{\pi \otimes i\}(\delta(a))(1 \otimes M_G(F))$, where $a \in A$ and $F \in C_0(G/H)$. Now this makes sense whether or not the subgroup H is normal or amenable. Using this subalgebra it may be possible to extend the results of this paper to include, at least, the non-normal case.

3. The Subalgebras

In this section H will be a closed (not necessarily normal or amenable) subgroup of G. For each such H we present a dense $*$-subalgebra $\mathscr{D}_H$ of $A \times_\delta (G/H)$. If H is the trivial subgroup we will denote $\mathscr{D}_H$ by $\mathscr{D}$. In the next section we will show that $\mathscr{D}$ is a pre-Hermitian $\mathscr{D}_H$-rigged $\mathscr{D}$ module. This will establish the desired induction procedure.

Suppose E is a compact subset of G, that $C_E(G)$ denotes the elements of $C_c(G)$ with support in E and that $u \in A_c(G)$. Then an element x of $B(\mathscr{H} \otimes L^2(G))$ is said to be (u, E, H) if it can be written as the norm limit of a sequence $(x_j)_{j=1}^\infty$ in $B(\mathscr{H} \otimes L^2(G))$ of the form

$$x_j = \sum_{i=1}^{n_j} \{\pi \otimes i\}(\delta(\delta_u(a_{ij})))(1 \otimes M_G(\varphi(f_{ij}))),$$

where the $f_{ij} \in C_E(G)$. We will denote by $\mathscr{D}_H$ the set of elements of $B(\mathscr{H} \otimes L^2(G))$ which are (u, E, H) for some $u \in A_c(G)$ and E compact in G. If H is the trivial subgroup we will abbreviate (u, E, H) by (u, E).

Proposition 8. *Suppose δ is a coaction of G on A, H is a closed subgroup of G, $a \in A$, u, $v \in A_c(G)$, and $f \in C_c(G)$. Then the maps*

$$: s \to \delta(\delta_{u^s \cdot v}(a))(1 \otimes \varphi(f_s)) \qquad \textit{and} \qquad : s \to (1 \otimes \varphi(f_s))\, \delta(\delta_{u_s \cdot v}(a))$$

are continuous with compact support, and

$$(1 \otimes \varphi(u * f))\, \delta(\delta_v(a)) = \int_G \delta(\delta_{u^s \cdot v}(a))(1 \otimes \varphi(f_s))\, ds$$

$$\delta(\delta_v(a))(1 \otimes \varphi(\check{u} * f)) = \int_G (1 \otimes \varphi(f_s))\, \delta(\delta_{u_s \cdot v}(a))\, ds.$$

Proof. Let $\gamma(s)=\delta(\delta_{u^s\cdot v}(a))(1\otimes\varphi(f_s))$. Then γ is clearly supported in the compact set $F=(\operatorname{supp} v)^{-1}(\operatorname{supp} u)$. Now

$$\begin{aligned}\|\gamma(s)-\gamma(s')\| \leqslant \|v\|_{B(G)}\,\|a\|_A(\|u^s-u^{s'}\|_{B(G)}\,\|\varphi(f)\|_{C_0(G/H)}\\ +\|u\|_{B(G)}\,\|(\varphi(f))_{sH}-(\varphi(f))_{s'H}\|_{C_0(G/H)}).\end{aligned}$$

This, the continuity of: $G\to A(G)$: $s\to u^s$, and the uniform continuity of $\varphi(f)$ implies γ is continuous. That : $s\to(1\otimes\varphi(f_s))\,\delta(\delta_{u_s\cdot v}(a))$ is also continuous with compact support follows similarly. In order to establish the first equation of the proposition we initially need to show that if $g\in C_c(G)$, then

$$M_G(\varphi(u*f))\,\lambda_G(g)=\int_G \mathscr{S}_{u^s}(\delta_G(\lambda_G(g)))\,M_G(\varphi(f_s))\,ds. \tag{2}$$

Let ξ, $\eta\in L^2(G)$. Then

$$\begin{aligned}
&\langle\{M_G(\varphi(u*f))\,\lambda_G(g)\}(\xi),\eta\rangle_{L^2(G)}\\
&=\int_G \{\{M_G(\varphi(u*f))\,\lambda_G(g)\}(\xi)\}\,(p)\,\overline{\eta(p)}\,dp\\
&=\int_G\int_G\int_H\int_G u(r)\,f(r^{-1}ph)\,g(t)\,\xi(t^{-1}p)\,\overline{\eta(p)}\,dr\,dh\,dt\,dp\\
&=\int_G\int_G\int_H\int_G u(ts)\,f(s^{-1}t^{-1}ph)g(t)\,\xi(t^{-1}p)\,\overline{\eta(p)}\,ds\,dh\,dt\,dp\\
&=\int_G\int_G\int_G g(t)\,u(ts)\,\xi(t^{-1}p)\left(\int_H f(s^{-1}t^{-1}ph)\,dh\right)\overline{\eta(p)}\,dp\,ds\,dt
\end{aligned}$$

(by Fubini's theorem since

$$\begin{aligned}
&\int_G\int_G\int_G\int_H |u(ts)\,f(s^{-1}t^{-1}ph)\,g(t)\,\xi(t^{-1}p)\,\overline{\eta(p)}|\,dh\,ds\,dp\,dt\\
&\qquad\leqslant \mu_G(E)\,\|u\|_{C_0(G)}\,\|\varphi(|f|)\|_{C_0(G/H)}\int_G(|g|\cdot(|\eta|*|\xi|^{\vee}))(t)\,dt<\infty,
\end{aligned}$$

where $E=(\operatorname{supp} g)^{-1}(\operatorname{supp} u)$; note that $|\eta|*|\xi|^{\vee}\in A(G)$ [4, Proposition 3.4])

$$=\left\langle\left\{\int_G\int_G g(t)\,u(ts)\,\lambda_G(t)\,M_G(\varphi(f_s))\,dt\,ds\right\}(\xi),\eta\right\rangle_{L^2(G)}$$

Since ξ and η were arbitrary we have that

$$\begin{aligned}
M_G(\varphi(u*f))\,\lambda_G(g) &= \int_G \int_G g(t)\,u(ts)\,\lambda_G(t)\,M_G(\varphi(f_s))\,dt\,ds \\
&= \int_G \int_G g(t)\,\mathcal{S}_{u^s}(\lambda_G(t)\otimes\lambda_G(t))\,M_G(\varphi(f_s))\,dt\,ds \\
&= \int_G \mathcal{S}_{u^s}\left(\int_G g(t)(\lambda_G(t)\otimes\lambda_G(t))\,dt\right)M_G(\varphi(f_s))\,ds \\
&= \int_G \mathcal{S}_{u^s}(\delta_G(\lambda_G(g)))\,M_G(\varphi(f_s))\,ds,
\end{aligned}$$

which establishes (2). By Lemma 1(i) we can find $a_{ij}\in A$ and $g_{ij}\in C_c(G)$ with supp $g_{ij}\subset$ supp v such that $\{\pi\otimes i\}(\delta(\delta_v(a)))$ is the $*$-strong limit of the net $\sum_{j=1}^{n_i}\pi(a_{ij})\otimes\lambda_G(g_{ij})$. Let

$$\gamma_i(s)=\{\pi\otimes\mathcal{S}_{u^s}\}\left(\{i\otimes\delta_G\}\left(\sum_{j=1}^{n_i}a_{ij}\otimes\lambda_G(g_{ij})\right)\right)(1\otimes M_G(\varphi(f_s))).$$

Then

$$\begin{aligned}
\gamma_i(s) &\to \{\pi\otimes\mathcal{S}_{u^s}\}(\{i\otimes\delta_G\}(\delta(\delta_v(a))))(1\otimes M_G(\varphi(f_s))) \\
&= \{\pi\otimes\mathcal{S}_{u^s}\}(\{\delta\otimes i\}(\delta(\delta_v(a))))(1\otimes M_G(\varphi(f_s))) \\
&= \{\pi\otimes i\}(\delta(\delta_{u^s\cdot v}(a)))(1\otimes M_G(\varphi(f_s))) \\
&= \gamma(s),
\end{aligned}$$

where the convergence is in the $*$-strong topology. A similar inequality to that from which we deduced the continuity of γ shows the γ_i are continuous. Each is compactly supported in $F=(\text{supp } v)^{-1}(\text{supp } u)$ as can be seen from

$$\begin{aligned}
\gamma_i(s) &= \sum_{j-1}^{n_i}\pi(a_{ij})\otimes\left(\mathcal{S}_{u^s}\left(\int_G g_{ij}(t)\,\lambda_G(t)\otimes\lambda_G(t)\,dt\right)(1\otimes M_G(\varphi(f_s)))\right) \\
&= \sum_{j=1}^{n_i}\pi(a_{ij})\otimes\left(\left(\int_G g_{ij}(t)\,u(ts)\,\lambda_G(t)\,dt\right)(1\otimes M_G(\varphi(f_s)))\right).
\end{aligned}$$

Let ξ, $\eta\in\mathcal{H}\otimes L^2(G)$. Let $\omega_{\xi,\eta}$ be the linear functional on $B(\mathcal{H}\otimes L^2(G))$ defined by $T\to\langle T(\xi),\eta\rangle$. Then

$$\begin{aligned}
&\omega_{\xi,\eta}((1\otimes M_G(\varphi(u*f)))\{\pi\otimes i\}(\delta(\delta_v(a))))\\
&\quad=\lim_{i\to\infty}\sum_{j=1}^{n_i}\omega_{\xi,\eta}(\pi(a_{ij})\otimes(M_G(\varphi(u*f))\,\lambda_G(g_{ij})))\\
&\quad=\lim_{i\to\infty}\sum_{j=1}^{n_i}\omega_{\xi,\eta}\left(\pi(a_{ij})\otimes\left(\int_G\mathscr{S}_{u^s}(\delta_G(\lambda_G(g_{ij})))\,M_G(\varphi(f_s))\,ds\right)\right)\quad\text{(by (2))}\\
&\quad=\lim_{i\to\infty}\int_G\omega_{\xi,\eta}\left(\{\pi\otimes\mathscr{S}_{u^s}\}\left(\{i\otimes\delta_G\}\right.\right.\\
&\qquad\qquad\left.\left.\left(\sum_{j=1}^{n_i}a_{ij}\otimes\lambda_G(g_{ij})\right)\right)(1\otimes M_G(\varphi(f_s)))\right)ds\\
&\quad=\int_G\omega_{\xi,\eta}(\{\pi\otimes i\}(\delta(\delta_{u^s\cdot v}(a)))(1\otimes M_G(\varphi(f_s))))\,ds
\end{aligned}$$

(by the dominated convergence theorem, since the $*$-strong convergence of the net $\sum_{j=1}^{n_i}\pi(a_{ij})\otimes\lambda(g_{ij})$ to $\{\pi\otimes i\}(\delta(\delta_v(a)))$ implies

$$\begin{aligned}
|\omega_{\xi,\eta}(\gamma_i(s))|\leqslant\|u\|_{B(G)}\,\|\varphi(f)\|_{C_0(G)}\,\|\xi\|&\\
\times(\|\{\{\pi\otimes i\}\,\delta(\delta_v(a))\}^*\,(\eta)\|+1)\,\chi_F(s)&
\end{aligned}$$

for i sufficiently large, where χ_F is the characteristic function of F)

$$=\omega_{\xi,\eta}\left(\int_G\{\pi\otimes i\}(\delta(\delta_{u^s\cdot v}(a)))(1\otimes M_G(\varphi(f_s)))\,ds\right).$$

This proves the first equation of the proposition. The second equation follows similarly. ∎

LEMMA 9. *Suppose δ is a coaction of G on A, H is a closed subgroup of G, $a\in A$, E is a compact subset of G, and $v\in A_c(G)$. Then*

(i) *there exists a compact subset F of G such that if $f\in C_E(G)$ and $\varepsilon>0$, then there exists $a_j\in A$ and $f_j\in C_F(G)$, for $j=1,\ldots,n$, such that*

$$\left\|(1\otimes\varphi(f))\,\delta(\delta_v(a))-\sum_{j=1}^{n}\delta(\delta_v(a_j))(1\otimes\varphi(f_j))\right\|<\varepsilon;$$

(ii) *there exists a compact subset F' of G such that if $f\in C_E(G)$ and $\varepsilon>0$, then there exist $a_j'\in A$ and $f_j'\in C_{F'}(G)$, for $j=1,\ldots,n'$, such that*

$$\left\|\delta(\delta_v(a))(1\otimes\varphi(f))-\sum_{j=1}^{n'}(1\otimes\varphi(f_j'))\,\delta(\delta_v(a_j'))\right\|<\varepsilon.$$

Proof. Suppose E is a compact subset of G, $f\in C_E(G)$, and V is a compact neighbourhood of the identity. Let F be the compact subset $(\operatorname{supp} v)^{-1} VE$ and let $\varepsilon > 0$. Choose $u \in A_c^+(G)$ such that $\|\varphi(u * f) - \varphi(f)\|_{C_0(G/H)} < \varepsilon/(2\,\|\delta_v(a)\|_A)$ and $\operatorname{supp} u \subset V$. Then

$$\|(1 \otimes \varphi(f))\, \delta(\delta_v(a)) - (1 \otimes \varphi(u * f))\, \delta(\delta_v(a))\| < \varepsilon/2.$$

By Proposition 8, $\gamma\colon s \to \delta(\delta_{u^s \cdot v}(a))(1 \otimes \varphi(f_s))$ is continuous and compactly supported in $(\operatorname{supp} v)^{-1} (\operatorname{supp} u) \subset (\operatorname{supp} v)^{-1} V$. So by, for example, [26, Proposition IV.7.3], we can find $\zeta_j \in C_c(G)$, also supported in $(\operatorname{supp} v)^{-1} V$ and $s_j \in G$, for $j = 1, ..., n$, such that $\|\gamma(s) - \sum_{j=1}^n \zeta_j(s)\, \gamma(s_j)\| < \varepsilon/(2\mu_G((\operatorname{supp} v)^{-1} V))$ for all $s \in G$. If we let $v_j = \int_G \zeta_j(s)\, ds$, then

$$\left\| \int_G \gamma(s)\, ds - \sum_{j=1}^{n} v_j \gamma(s_j) \right\| \leqslant \int_G \left\| \gamma(s) - \sum_{j=1}^{n} \zeta_j(s)\, \gamma(s_j) \right\| ds < \varepsilon/2.$$

By Proposition 8, $(1 \otimes \varphi(u * f))\, \delta(\delta_v(a)) = \int_G \delta(\delta_{u^s \cdot v}(a))(1 \otimes \varphi(f_s))\, ds$. Combining this and the above facts gives

$$\begin{aligned}
&\left\| (1 \otimes \varphi(f))\, \delta(\delta_v(a)) - \sum_{j=1}^{n} \delta(\delta_v(\delta_{u^{s_j}}(a)))(1 \otimes \varphi(v_j f_{s_j})) \right\| \\
&\quad \leqslant \|(1 \otimes \varphi(f))\, \delta(\delta_v(a)) - (1 \otimes \varphi(u * f))\, \delta(\delta_v(a))\| \\
&\qquad + \left\| (1 \otimes \varphi(u * f))\, \delta(\delta_v(a)) - \sum_{j=1}^{n} v_j\, \delta(\delta_{u^{s_j} \cdot v}(a))(1 \otimes \varphi(f_{s_j})) \right\|
\end{aligned}$$

(since $\delta_v(\delta_{u^{s_j}}(a)) = \delta_{v \cdot u^{s_j}}(a) = \delta_{u^{s_j} \cdot v}(a)$ by Lemma 1)

$$< \varepsilon/2 + \left\| \int_G \gamma(s)\, ds - \sum_{j=1}^{n} v_j \gamma(s_j) \right\| < \varepsilon.$$

Choosing $a_j = \delta_{u^{s_j}}(a)$ and $f_j = v_j f_{s_j}$ gives the lemma since $f_j = v_j f_{s_j} \in C_F(G)$. Part (ii) follows similarly. ∎

LEMMA 10. *Let E and F be compact subsets of G. Let u and $v \in A_c(G)$. Then*

(i) *there exists a compact subset D of G and $w \in A_c(G)$ such that if $f \in C_E(G)$ and $g \in C_F(G)$, then $\delta(\delta_u(a))(1 \otimes \varphi(f)) + \delta(\delta_v(b))(1 \otimes \varphi(g))$ is (w, D, H);*

(ii) *there exists a compact subset D' of G such that if $f \in C_E(G)$, then $(\delta(\delta_u(a))(1 \otimes \varphi(f)))^*$ is $(\tilde{u}, D', H)$;*

(iii) *there exists $w' \in A_c(G)$ such that if $g \in C_E(G)$ and $f \in C_c(G)$, then $\delta(\delta_u(a))(1 \otimes \varphi(f))\, \delta(\delta_v(b))(1 \otimes \varphi(g))$ is (w', E, H);*

(iv) *there exists $w'' \in A_c(G)$ such that if $g \in C_E(G)$ and $f \in C_c(G)$, then $\delta(\delta_u(a))(1 \otimes \varphi(f))\, \delta(\delta_v(b))(1 \otimes g)$ is (w'', E);*

(v) *there exists* $w''' \in A_c(G)$ *and a compact subset* D'' *of* G *such that if* $f \in C_E(G)$ *and* $g \in C_c(G)$, *then* $\delta(\delta_u(a))(1 \otimes f)\, \delta(\delta_v(b))(1 \otimes \varphi(g))$ *is* (w''', D'').

Proof. (i) Let $D = E \cup F$. Then D is compact. Let $w \in A_c(G)$ be such that w restricted to $(\operatorname{supp} u) \cup (\operatorname{supp} v)$ is identically one. Then $\delta_w(\delta_u(a)) = \delta_u(a)$ and $\delta_w(\delta_v(b)) = \delta_v(b)$ (Lemma 1), so if $f \in C_E(G)$ and $g \in C_F(G)$, then

$$\delta(\delta_u(a))(1 \otimes \varphi(f)) + \delta(\delta_v(b))(1 \otimes \varphi(g)) = \sum_{i=1}^{2} \delta(\delta_w(e_i))(1 \otimes \varphi(c_i)),$$

where $e_1 = a$, $e_2 = b$ with the $e_i \in A$, and $c_1 = f$, $c_2 = g$ with the $c_i \in C_D(G)$. So $\delta(\delta_u(a))(1 \otimes \varphi(f)) + \delta(\delta_v(b))(1 \otimes \varphi(g))$ is (w, D, H).

(ii) By Lemma 9 there exists a compact subset D' of G such that if $f \in C_E(G)$ and $\varepsilon > 0$, then there exist $f_j \in C_{D'}(G)$ and $a_j \in A$ such that

$$\left\| \delta(\delta_u(a))(1 \otimes \varphi(f)) - \sum_{j=1}^{n} (1 \otimes \varphi(f_j))\, \delta(\delta_u(a_j)) \right\| < \varepsilon.$$

Hence

$$\left\| (\delta(\delta_u(a))(1 \otimes \varphi(f)))^* - \sum_{j=1}^{n} \delta(\delta_{\tilde{u}}(a_j^*))(1 \otimes \varphi(\bar{f}_j)) \right\| < \varepsilon,$$

(since $(\delta_u(a_j))^* = \delta_{\tilde{u}}(a_j^*)$ by Lemma 1). Thus if $f \in C_E(G)$, then $(\delta(\delta_u(a))(1 \otimes \varphi(f)))^*$ is $(\tilde{u}, D', H)$.

(iii) Let $\varepsilon > 0$. By Lemma 9 there exist $b_j \in A$ and $f_j \in C_c(G)$ such that

$$\left\| (1 \otimes \varphi(f))\, \delta(\delta_v(b)) - \sum_{j=1}^{n} \delta(\delta_v(b_j))(1 \otimes \varphi(f_j)) \right\|$$
$$< \varepsilon / (\|\delta_u(a)\|_A \, \|\varphi(g)\|_{C_0(G/H)}).$$

Let $w' \in A_c(G)$ be one on $(\operatorname{supp} u)(\operatorname{supp} v)$. Then $\delta_{w'}(\delta_u(a)\, \delta_v(b_j)) = \delta_u(a)\, \delta_v(b_j)$ (Lemma 1). Also $(\varphi(f_j)) \cdot (\varphi(g)) = \varphi(\varphi(f_j) \cdot g)$, so

$$\begin{aligned}
&\Big\| \delta(\delta_u(a))(1 \otimes \varphi(f))\, \delta(\delta_v(b))(1 \otimes \varphi(g)) \\
&\qquad - \sum_{j=1}^{n} \delta(\delta_{w'}(\delta_u(a)\, \delta_v(b_j)))(1 \otimes \varphi(\varphi(f_j) \cdot g)) \Big\| \\
&= \Big\| \delta(\delta_u(a)) \Big((1 \otimes \varphi(f))\, \delta(\delta_v(b)) \\
&\qquad - \sum_{j=1}^{n} \delta(\delta_v(b_j))(1 \otimes \varphi(f_j)) \Big) (1 \otimes \varphi(g)) \Big\| < \varepsilon.
\end{aligned}$$

Thus if $g \in C_E(G)$, then $\delta(\delta_u(a))(1 \otimes \varphi(f))\, \delta(\delta_v(b))(1 \otimes \varphi(g))$ is (w', E, H).

(iv) and (v) follow similarly. ∎

LEMMA 11. *Suppose E and F are compact subsets of G, u, $v \in A_c(G)$, x, $y \in \mathscr{D}_H$, and $z \in \mathscr{D}$.*

(i) *If x and y are (u, E, H) and (v, F, H), respectively, then there exists a compact subset D of G and $w \in A_c(G)$, such that $x + y$ is (w, D, H). Hence $\mathscr{D}_H$ is closed under addition.*

(ii) *If x is (u, E, H), then there exists a compact subset D' of G such that x^* is $(\tilde{u}, D', H)$. Hence $\mathscr{D}_H$ is closed under adjoints.*

(iii) *If y is (v, E, H), then there exists $w \in A_c(G)$, such that xy is (w, E, H). Hence $\mathscr{D}_H$ is closed under multiplication.*

(iv) *If z is (v, E), then there exists $w' \in A_c(G)$, such that xz is (w', E) and hence $xz \in \mathscr{D}$.*

(v) *If z is (v, E), then there exists a compact subset D'' of G and $w'' \in A_c(G)$ such that zx is (w'', D'') and hence $zx \in \mathscr{D}$.*

(vi) *If P is a compact subset of H, and z is (u, E), then there exists a compact subset D''' of G such that $\hat{\delta}_h(x)$ is (u, D''') for all $h \in P$, where $\hat{\delta}$ is the dual action of G on $A \times_\delta G$ (restricted to H).*

Proof. By assumption

$$x = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij}))$$

and

$$y = \lim_{k \to \infty} \sum_{l=1}^{m_k} \delta(\delta_v(b_{kl}))(1 \otimes \varphi(g_{kl})),$$

for some a_{ij}, $b_{kl} \in A$ and f_{ij}, $g_{kl} \in C_c(G)$.

(i) Since x and y are (u, E, H) and (v, F, H), respectively, the f_{ij} and g_{kl} are elements of $C_E(G)$ and $C_F(G)$, respectively. By the continuity of addition

$$x + y = \lim_{i \to \infty} \left(\sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij})) + \sum_{l=1}^{m_i} \delta(\delta_v(b_{il}))(1 \otimes \varphi(g_{il})) \right).$$

By Lemma 10 there exists a compact subset D of G and $w \in A_c(G)$ such that $\delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij})) + \delta(\delta_u(b_{il}))(1 \otimes \varphi(g_{il}))$ is (w, D, H) for all i, j, and l. Since a finite sum of (w, D, H) elements is (w, D, H), each term in the limit is (w, D, H). Hence the limit, that is, $x + y$, is (w, D, H).

(ii) Since x is (u, E, H), the f_{ij} are elements of $C_E(G)$. By the continuity of the adjoint operation

$$x^* = \lim_{i \to \infty} \sum_{j=1}^{n_i} (\delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij})))^*.$$

By Lemma 10 there exists a compact subset D' such that $(\delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij})))^*$ is $(\tilde{u}, D', H)$ for all i and j. Since a finite sum of $(\tilde{u}, D', H)$ elements is $(\tilde{u}, D', H)$, each term in the limit is $(\tilde{u}, D', H)$. Hence the limit, that is, x^*, is $(\tilde{u}, D', H)$.

(iii)–(v) Follow similarly.

(vi) Now by assumption $z = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij})$ for some $a_{ij} \in A$ and $f_{ij} \in C_E(G)$. By the continuity of $\hat{\delta}_h$

$$\begin{aligned}\hat{\delta}_h(z) &= \lim_{i \to \infty} \sum_{j=1}^{n_i} \hat{\delta}_h(\delta(\delta_u(a_{ij}))(1 \otimes f_{ij})) \\ &= \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij}^h),\end{aligned}$$

(since $\hat{\delta}_h(\delta(a)) = \delta(a)$ and $\hat{\delta}_h(1 \otimes f) = 1 \otimes f^h$), where $\operatorname{supp} f_{ij}^h \subset EP^{-1}$ for all i and j, hence $\hat{\delta}_h(z)$ is (u, EP^{-1}) for all $h \in P$. So choose D''' to be EP^{-1}. ∎

THEOREM 12. *Suppose δ is a non-degenerate coaction of G on A, and H is a closed subgroup of G. Then $\mathscr{D}_H$ is a dense $*$-subalgebra of $A \times_\delta (G/H)$.*

Proof. By Lemma 11, $\mathscr{D}_H$ is closed under the algebraic operations and involution. So it remains to show $\mathscr{D}_H$ is dense in $A \times_\delta (G/H)$. Since δ is non-degenerate the set $\{\delta_u(a)\colon a \in A,\ u \in A_c(G)\}$ is dense in A (see [4, Proposition 3.4; 12, Theorem 5]). From this it is clear that the linear span of the elements $\delta(\delta_u(a))(1 \otimes \varphi(f))$ for $u \in A_c(G)$, $a \in A$, and $f \in C_c(G)$, and hence $\mathscr{D}_H$, is dense in $A \times_\delta (G/H)$. ∎

Suppose E is a compact subset of G and $u \in A_c(G)$. We will say a continuous, compactly supported function $\xi\colon H \to \mathscr{D}$ is (u, E) if $\xi(h)$ is (u, E) for all $h \in H$. We will denote by $\mathscr{I}_H$ the set of all elements of $C_c(H, \mathscr{D})$ which are (u, E) for some $u \in A_c(G)$ and E compact in G.

PROPOSITION 13. *Suppose δ is a non-degenerate coaction of G on A and H is a closed subgroup of G. Then $\mathscr{I}_H$ is a dense $*$-subalgebra of $(A \times_\delta G) \times_{\hat{\delta}} H$.*

Proof. Let $\xi, \gamma \in \mathscr{I}_H$, with ξ being (u, E) and γ being (v, F), where E, F are compact subsets of G and $u, v \in A_c(G)$.

(i) Now $\xi(h)$ is (u, E) and $\gamma(h)$ is (v, F) for all $h \in H$, so by Lemma 11(i) there exists a compact subset D of G and $w \in A_c(G)$ such that $(\xi+\gamma)(h)$ is (w, D) for all $h \in H$; that is, $\xi+\gamma$ is (w, D). Clearly $\xi+\gamma$ is continuous and compactly supported so $\xi+\gamma \in \mathscr{I}_H$ and $\mathscr{I}_H$ is closed under addition.

(ii) Now $\xi(h)$ is (u, E) for all $h \in H$, so by Lemma 11(ii) there exists a compact subset D of G such that $\xi(h^{-1})^*$ is $(\tilde{u}, D)$ for all $h \in H$. By Lemma 11(vi) there exists a compact subset L of G such that $\xi^*(h) = (1/\Delta h)\, \hat{\delta}_h(\xi(h^{-1})^*)$ is $(\tilde{u}, L)$ for all $h \in (\operatorname{supp} \xi)^{-1}$, hence for all $h \in H$. Now ξ^* is continuous and compactly supported so $\xi^* \in \mathscr{I}_H$ and $\mathscr{I}_H$ is self adjoint.

(iii) Now $\xi(h)$ is (u, E) and $\gamma(h)$ is (v, F) for all $h \in H$, so by part (vi) of Lemma 11 there exists a compact subset D of G such that $\hat{\delta}_r(\gamma(r^{-1}h))$ is (v, D), for all $r \in \operatorname{supp} \xi$, $h \in H$, and hence for all $r, h \in H$. By Lemma 11(iii) there exists $w \in A_c(G)$ such that $\xi(r)\, \hat{\delta}_r(\gamma(r^{-1}h))$ is (w, D) for all $r, h \in H$. Now $\zeta_h \colon r \to \xi(r)\, \hat{\delta}_r(\gamma(r^{-1}h))$ is continuous and compactly supported so we can find $r_{ij}^h \in \operatorname{supp} \xi$, $\gamma_{ij}^h \in C_c(H)$ such that ζ_h is the uniform limit of the $\sum_{j=1}^{n_i} \gamma_{ij}^h \xi(r_{ij}^h)$ (see, for example, [26, Proposition IV.7.3]). Hence

$$\{\xi * \gamma\}(h) = \int_H \zeta_h(r)\, dr = \lim_{i \to \infty} \sum_{j=1}^{n_i} \left(\int_H \gamma_{ij}^h(r)\, dr \right) \xi(r_{ij}^h).$$

Now since each term in the limit is (w, D), $\{\xi * \gamma\}(h)$ is (w, D) for all $h \in H$. Since $\xi * \gamma$ is continuous and compactly supported $\xi * \gamma \in \mathscr{I}_H$ and $\mathscr{I}_H$ is closed under multiplication.

(iv) Since the maps: $h \to \sum_{i=1}^n \eta_i(h) x_i$ are in $\mathscr{I}_H$, where the $\eta_i \in C_c(H)$ and the $x_i \in \mathscr{D}$, $\mathscr{I}_H$ is dense in $C_c(H, A \times_\delta G)$, and hence in $(A \times_\delta G) \times_{\hat{\delta}} H$. ∎

4. Induced Representations of Crossed Products by Coactions

In this section we show how representations of $A \times_\delta G$ can be constructed from those of $A \times_{\delta|} (G/H)$. We will acheive this by showing that $\mathscr{D}$ is a pre-Hermitian $\mathscr{D}_H$-rigged $\mathscr{D}$-module, to which we can apply Rieffel's theory on the induction of representations of C^*-algebras [21].

Throughout this section δ will be a non-degenerate coaction of G on A and H will be a closed normal amenable subgroup of G. We define a right action of $\mathscr{D}_H$ on $\mathscr{D}$ by $x \bullet z = xz$, for $x \in \mathscr{D}$ and $z \in \mathscr{D}_H$. Note that the action is well defined by Lemma 11(iv). Our immediate goal is to show that $\mathscr{D}$ is a $\mathscr{D}_H$-rigged space. In order to define a pre-$\mathscr{D}_H$-valued inner product on $\mathscr{D}$ we need to establish the following results.

LEMMA 14. *Suppose $\mathscr{H}$ is a Hilbert space, $\gamma_i \in \mathscr{H}$, for $i = 1, \ldots, n$, E is a compact subset of G, and $\omega_E \in C_c^+(G)$ is one on E. Then there exists a positive constant α_E such that*

(i) *if $\xi_i \in C_E(G)$, for $i = 1, \ldots, n$, then*

$$\left\| \sum_{i=1}^{n} \gamma_i \otimes \varphi(\xi_i) \right\|_{\mathscr{H} \otimes L^2(G/H)} \leqslant \alpha_E \left\| \sum_{i=1}^{n} \gamma_i \otimes \xi_i \right\|_{\mathscr{H} \otimes L^2(G)},$$

(ii) *if $\eta_i \in C_c(G/H)$, for $i = 1, \ldots, n$, then*

$$\left\| \sum_{i=1}^{n} \gamma_i \otimes (\omega_E \cdot \eta_i) \right\|_{\mathscr{H} \otimes L^2(G)} \leqslant \alpha_E \left\| \sum_{i=1}^{n} \gamma_i \otimes \eta_i \right\|_{\mathscr{H} \otimes L^2(G/H)}$$

Proof. First we show two inequalities that we will need later in the proof. Suppose $\xi \in C_E(G)$ and $\eta \in C_c(G/H)$. Then by Hölders inequality

$$\begin{aligned} \|\varphi(\xi)\|^2_{L^2(G/H)} &\leqslant \int_{G/H} \left(\int_H \omega_E(sh)\, |\xi(sh)|\, dh \right)^2 dsH \\ &\leqslant \int_{G/H} \int_H \omega_E(sr)^2\, dr \int_H |\xi(sh)|^2\, dh\, dsH \\ &\leqslant \|\varphi(\omega_E^2)\|_{C_0(G/H)}\, \|\xi\|_{L^2(G)}. \end{aligned} \tag{3}$$

Also

$$\begin{aligned} \|\omega_E \cdot \eta\|^2_{L^2(G)} &= \int_{G/H} \int_H \omega_E^2(sh)\, |\eta(sh)|^2\, dh\, dsH \\ &= \int_{G/H} \left(\int_H \omega_E^2(sh)\, dh \right) |\eta(sH)|^2\, dsH \\ &\leqslant \|\varphi(\omega_E^2)\|_{C_0(G/H)}\, \|\eta\|^2_{L^2(G/H)}. \end{aligned} \tag{4}$$

Suppose $\alpha_E = \|\varphi(\omega_E^2)\|_{C_0(G/H)}$, $(\varepsilon_j)_{j \in J}$ is an orthonormal basis of $\mathscr{H}$, and

$$\mathscr{F} = \left\{ \sum_{j \in J} v_j \varepsilon_j : v_j \in \mathbb{C} \text{ with all but finitely many } v_j = 0 \right\}.$$

If the $\gamma_i = \sum v_{ij} \varepsilon_j \in \mathscr{F}$ and $\xi_i \in C_E(G)$, for $i = 1, \ldots, n$, then

$$\begin{aligned} \left\| \sum_{i=1}^{n} \gamma_i \otimes \varphi(\xi_i) \right\|^2_{\mathscr{H} \otimes L^2(G/H)} &= \left\| \sum_{j \in J} \varepsilon_j \otimes \varphi\left(\sum_{i=1}^{n} v_{ij} \xi_i \right) \right\|^2_{\mathscr{H} \otimes L^2(G/H)} \\ &= \sum_{j \in J} \left\| \varphi\left(\sum_{i=1}^{n} v_{ij} \xi_i \right) \right\|^2_{\mathscr{H} \otimes L^2(G/H)} \end{aligned}$$

(by the orthonormality of the ε_j)

$$\leqslant \sum_{j \in J} \alpha_E \left\| \sum_{i=1}^{n} v_{ij} \xi_i \right\|^2_{\mathscr{H} \otimes L^2(G)}$$

(by (3) since for all j, $\sum v_{ij} \xi \in C_E(G)$)

$$= \alpha_E \left\| \sum_{i=1}^{n} \gamma_i \otimes \xi_i \right\|^2_{\mathscr{H} \otimes L^2(G)}$$

(by the orthonormality of the ε_j). Part (i) now follows easily from the density of $\mathscr{F}$ in $\mathscr{H}$. A similar argument using (4) gives (ii). ∎

LEMMA 15. *Suppose E is a compact subset of G and $v \in A_c(G)$. Then there exists a positive constant $\zeta_{v,E}$ such that if $a_i \in A$ and $f_i \in C_E(G)$, for $i = 1, ..., n$, then*

$$\left\| \sum_{i=1}^{n} \delta| (\delta_v(a_i))(1 \otimes \varphi(f_i)) \right\|_{A \times_{\delta|} (G/H)} \leqslant \zeta_{v,E} \left\| \sum_{i=1}^{n} \delta(\delta_v(a_i))(1 \otimes f_i) \right\|_{A \times_\delta G}.$$

Proof. By Lemma 1 we can find $a^k_{ij} \in A$ and $g^k_{ij} \in C_c(G)$ with supp $g^k_{ij} \subset$ supp v such that $\delta(\delta_v(a_i))$ is the strict limit of the net $(\sum_{j=1}^{m_{ik}} a^k_{ij} \otimes \lambda_G(g^k_{ij}))$ in $M(A \otimes C^*_r(G))$. Let $\sum_{l=1}^{p} \gamma_l \otimes \xi_l \in \mathscr{H} \otimes C_c(G/H)$. Then

$$\left\| \left\{ \sum_{i=1}^{n} \{\pi \otimes i\}(\delta| (\delta_v(a_i)))(1 \otimes M_{G/H}(\varphi f_i)) \right\} \left(\sum_{l=1}^{p} \gamma_l \otimes \xi_l \right) \right\|_{\mathscr{H} \otimes L^2(G/H)}$$

$$= \lim_{k \to \infty} \left\| \left\{ \sum_{i,j} \pi(a^k_{ij}) \otimes (\lambda_{G/H}(\varphi g^k_{ij})\, M_{G/H}(\varphi f_i)) \right\} \left(\sum_{l=1}^{p} \gamma_l \otimes \xi_l \right) \right\|_{\mathscr{H} \otimes L^2(G/H)}$$

$$= \lim_{k \to \infty} \left\| \sum_{i,j,l} (\{\pi(a^k_{ij})\}(\gamma_l)) \otimes (\varphi(\{\lambda_G(g^k_{ij})\, M_G(f_i)\}(\omega_E \cdot \xi_l))) \right\|_{\mathscr{H} \otimes L^2(G/H)}$$

(by a straightforward computation, where $\omega_E \in C^+_c(G)$ is one on E)

$$\leqslant \alpha_F \lim_{k \to \infty} \left\| \left\{ \sum_{i,j} \pi(a^k_{ij}) \otimes (\lambda_G(g^k_{ij})\, M_G(f_i)) \right\} \left(\sum_{l=1}^{p} \gamma_l \otimes (\omega_E \cdot \xi_l) \right) \right\|_{\mathscr{H} \otimes L^2(G)}$$

(where F is the compact set $(\operatorname{supp} v)E$ and α_F is the positive constant given by Lemma 14(i); note that Lemma 14 applies since for all i, j, k, and l the support of $\{\lambda_G(g^k_{ij})\, M_G(f_i)\}(\omega_E \cdot \xi_l)$ is contained in F)

$$= \alpha_F \left\| \left\{ \sum_{i=1}^{n} \{\pi \otimes i\}(\delta(\delta_v(a_i)))(1 \otimes M_G(f_i)) \right\} \left(\sum_{l=1}^{p} \gamma_l \otimes (\omega_E \cdot \xi_l) \right) \right\|_{\mathscr{H} \otimes L^2(G)}.$$

We use this below. Define $\psi_E : \mathscr{H} \odot C_c(G/H) \to \mathscr{H} \odot C_c(G)$ by $\psi_E(\sum_{l=1}^p \gamma_l \otimes \xi_l) = \sum_{l=1}^p \gamma_l \otimes (\omega_E \cdot \xi_l)$. Then

$$\left\| \sum_{i=1}^n \{\pi \otimes i\}(\delta| (\delta_v(a_i)))(1 \otimes M_{G/H}(\varphi f_i)) \right\|_{A \times_{\delta|} (G/H)}$$

$$= \sup\left\{ \frac{\| \{\sum_{i=1}^n \{\pi \otimes i\}(\delta| (\delta_v(a_i)))(1 \otimes M_{G/H}(\varphi f_i))\}(\eta) \|_{\mathscr{H} \otimes L^2(G/H)}}{\|\eta\|_{\mathscr{H} \otimes L^2(G/H)}} : \eta \in \mathscr{H} \odot C_c(G/H) \right\}$$

(since $\mathscr{H} \odot C_c(G/H)$ is dense in $\mathscr{H} \otimes L^2(G/H)$)

$$\leqslant \alpha_F \sup\left\{ \frac{\| \{\sum_{i=1}^n \{\pi \otimes i\}(\delta(\delta_v(a_i)))(1 \otimes M_G(f_i))\}(\psi_E(\eta)) \|_{\mathscr{H} \otimes L^2(G)}}{\|\eta\|_{\mathscr{H} \otimes L^2(G/H)}} : \eta \in \mathscr{H} \odot C_c(G/H) \right\}$$

(by the above)

$$\leqslant \alpha_F \alpha_E \sup\left\{ \frac{\| \{\sum_{i=1}^n \{\pi \otimes i\}(\delta(\delta_v(a_i)))(1 \otimes M_G(f_i))\}(\psi_E(\eta)) \|_{\mathscr{H} \otimes L^2(G)}}{\|\psi_E(\eta)\|_{\mathscr{H} \otimes L^2(G)}} : \eta \in \mathscr{H} \odot C_c(G/H) \right\}$$

(by Lemma 14(ii))

$$\leqslant \alpha_F \alpha_E \left\| \sum_{i=1}^n \{\pi \otimes i\}(\delta(\delta_v(a_i)))(1 \otimes M_G(f_i)) \right\|_{A \times_\delta G}.$$

So choose $\zeta_{v,E} = \alpha_F \alpha_E$. ∎

PROPOSITION 16. *The map Ψ: $\mathscr{D} \to \mathscr{D}_H$ given by*

$$\lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij}) \to \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij}))$$

is well defined.

Proof. Suppose that x can be expressed as an element of $\mathscr{D}$ in the following ways

$$x = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij}) = \lim_{k \to \infty} \sum_{l=1}^{m_k} \delta(\delta_v(b_{kl}))(1 \otimes g_{kl}),$$

where a_{ij}, $b_{kl} \in A$, u, $v \in A_c(G)$, $f_{ij} \in C_D(G)$, and $g_{kl} \in C_E(G)$ for some fixed compact sets D and E of G. Then by Lemma 11(i) we can find $\omega \in A_c(G)$ and a compact subset F of G such that

$$\sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij}) - \sum_{l=1}^{m_k} \delta(\delta_v(b_{kl}))(1 \otimes g_{kl})$$

is (ω, F) for all i and k. So by Proposition 7 and Lemma 15 we have

$$\begin{aligned}
&\left\| \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij})) - \sum_{l=1}^{m_k} \delta(\delta_v(b_{kl}))(1 \otimes \varphi(g_{kl})) \right\|_{A \times_\delta (G/H)} \\
&\quad = \left\| \sum_{j=1}^{n_i} \delta| (\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij})) - \sum_{l=1}^{m_k} \delta| (\delta_v(b_{kl}))(1 \otimes \varphi(g_{kl})) \right\|_{A \times_{\delta|} (G/H)} \\
&\quad \leqslant \zeta_{\omega, F} \left\| \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij}) - \sum_{l=1}^{m_k} \delta(\delta_v(b_{kl}))(1 \otimes g_{kl}) \right\|_{A \times_\delta G}
\end{aligned}$$

for all i and k, where $\zeta_{\omega,F}$ is the constant of Lemma 15. Choosing the second expression for x the same as the first shows that the sequence $(\sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij})))$ converges. It is also clear from the inequality that $\Psi(x)$ is independent of how we express x. ∎

LEMMA 17. *Suppose that $(e_i)_{i \in I}$ is an (increasing) approximate identity of A, that $\psi \in A_c(G)$ is positive as an element of $C_r^*(G)^*$, with $\psi(1) = \|\psi\|_{B(G)} = 1$, and that $\mathscr{E}_H$ is the compact subsets of G/H ordered by inclusion. For each $E \in \mathscr{E}_H$ let ω_E denote an element of $C_c^*(G/H)$ which is one on E. Then*

(i) *$(\delta_\psi(e_i))_{i \in I}$ is an approximate identity of A,*

(ii) *the net $z_{(i,E)} = \delta(\delta_\psi(e_i))(1 \otimes \omega_E)$, for $(i, E) \in I \times \mathscr{E}_H$, converges strictly to 1 in $M(A \times_\delta (G/H))$ and hence converges to 1 $*$-strongly in $B(\mathscr{H} \otimes L^2(G))$.*

Proof. (i) Let p be a positive functional on A. Then $p(\delta_\psi(e_i)) = p \otimes \psi(\delta(e_i)) \geqslant 0$. Hence $\delta_\psi(e_i) \geqslant 0$. Similarly, we have that $i \leqslant j$ implies $\delta_\psi(e_i) \leqslant \delta_\psi(e_j)$. Also $\|\delta_\psi(e_i)\| \leqslant \|\psi\|_{B(G)} \|e_i\|_A \leqslant 1$. To see that $\delta_\psi(e_i) \to 1$ strictly in $M(A)$, write $\psi = b \bullet v$ for some $b \in C_r^*(G)$, $v \in B_r(G)$ and recall that $\delta(e_i) \to 1$ strictly in $M(A \otimes C_r^*(G))$. Then

$$\begin{aligned}
\delta_\psi(e_i)a &= \mathscr{S}_v(\delta(e_i)(a \otimes b)) \to \mathscr{S}_v(a \otimes b) \\
&= \mathscr{S}_\psi(a \otimes 1) = a\psi(1) = a \qquad \forall a \in A.
\end{aligned}$$

Similarly $a\delta_\psi(e_i) \to a$.

(ii) Let $\varepsilon > 0$ and $z \in A \times_\delta (G/H)$. Then by Theorem 12 we can find a_j, $b_k \in A$ and f_j, $g_k \in C_c(G)$ such that

$$\left\| z - \sum_{j=1}^{n} \delta(a_j)(1 \otimes \varphi(f_j)) \right\| < \varepsilon/5 \quad \text{and} \quad \left\| z^* - \sum_{k=1}^{m} \delta(b_k^*)(1 \otimes \varphi(\bar{g}_k)) \right\| < \varepsilon/5.$$

Choose i_0 such that $i \geqslant i_0$ implies $\|\delta_\psi(e_i)a_j - a_j\| < \varepsilon/(5rn)$ where r is the maximum of the $\|\varphi(f_j)\|_{C_0(G/H)}$. Let $E_0 = \bigcup_{k=1}^m \operatorname{supp} \varphi(g_k)$. Then for $(i, E) \geqslant (i_0, E_0)$

$$\begin{aligned}
\|z_{(i,E)}z - z\| \leqslant{} & \left\| \delta(\delta_\psi(e_i))(1 \otimes \omega_E) \left(z - \sum_{k=1}^{m} (1 \otimes \varphi(g_k))\, \delta(b_k) \right) \right\| \\
& + \left\| \delta(\delta_\psi(e_i)) \left(\sum_{k=1}^{m} (1 \otimes \varphi(g_k))\, \delta(b_k) - \sum_{j=1}^{n} \delta(a_j)(1 \otimes \varphi(f_j)) \right) \right\| \\
& + \sum_{j=1}^{n} \|\delta_\psi(e_i)a_j - a_j\| \, \|\varphi(f_j)\|_{C_0(G/H)} \\
& + \left\| \sum_{j=1}^{n} \delta(a_j)(1 \otimes \varphi(f_j)) - z \right\| \\
\leqslant{} & \varepsilon/5 + 2\varepsilon/5 + \varepsilon/5 + \varepsilon/5 = \varepsilon,
\end{aligned}$$

that is, $\|z_{(i,E)}z - z\| \to 0$. A similar argument shows $\|zz_{(i,E)} - z\| \to 0$. ∎

LEMMA 18. *Let $x, y \in \mathscr{D}$. Then the maps: $h \to \hat{\delta}_h(x)\, y$ and $:h \to y\hat{\delta}_h(x)$ are norm continuous with compact support, $\Psi(x) \in M(A \times_\delta G)$,*

$$\Psi(x)\, y = \int_H \hat{\delta}_h(x)\, y \, dh \qquad \text{and} \qquad y\Psi(x) = \int_H y\hat{\delta}_h(x)\, dh.$$

Proof. Let ζ: $H \to B(L^2(G))$ be defined by: $h \to \rho_H(h)\, M_G(f)\, \rho_H^*(h)$, where $f \in C_c(G)$. Let ξ, $\eta \in L^2(G)$. Let $\omega_{\xi,\eta}$ be the linear functional on $B(L^2(G))$ defined by $T \to \langle T(\xi), \eta \rangle_{L^2(G)}$. It is easily checked that $\omega_{\xi,\eta} \circ \zeta$ is Lebesgue integrable for all ξ and η. Now

$$\int_H \omega_{\xi,\eta}(\rho_H(h)\, M_G(f)\, \rho_H^*(h))\, dh = \omega_{\xi,\eta}(M_G(\varphi(f))).$$

So: $h \to \rho_H(h)\, M_G(f)\, \rho_H^*(h)$ is integrable with integral $M_G(\varphi(f))$ and

$$\begin{aligned}\int_H \hat{\delta}_h(1 \otimes M_G(f))\, dh &= \int_H 1 \otimes (\rho_H(h)\, M_G(f)\, \rho_H^*(h))\, dh \\ &= 1 \otimes \int_H \rho_H(h)\, M_G(f)\, \rho_H^*(h)\, dh \\ &= 1 \otimes M_G(\varphi(f)),\end{aligned}$$

or in abbreviated notation $\int_H \hat{\delta}_h(1 \otimes f)\, dh = 1 \otimes \varphi(f)$. By assumption there exist compact subsets E, F of G and u, $v \in A_c(G)$ such that

$$x = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij}) \quad \text{and} \quad y^* = \lim_{k \to \infty} \sum_{l=1}^{m_k} \delta(\delta_{\bar{v}}(b_{kl}^*))(1 \otimes \bar{g}_{kl}),$$

for some a_{ij}, $b_{kl} \in A$, $f_{ij} \in C_E(G)$, and $g_{kl} \in C_F(G)$. Let

$$\gamma_i(h) = \sum_{j,l} \delta(\delta_u(a_{ij}))(\hat{\delta}_h(1 \otimes f_{ij}))(1 \otimes g_{il})\, \delta(\delta_v(b_{il})).$$

Since $\hat{\delta}$ is an action of H on $B(\mathscr{H} \otimes L^2(G))$ the γ_i are norm continuous, and since $(\hat{\delta}_h(1 \otimes f_{ij}))(1 \otimes g_{il}) = 1 \otimes (f_{ij}^h \cdot g_{il})$ the γ_i are compactly supported in $(\operatorname{supp} g_{il})^{-1} (\operatorname{supp} f_{ij}) \subset FE^{-1}$. The map: $h \to \hat{\delta}_h(x) y$ is the limit of the γ_i so it is also compactly supported in FE^{-1}. It is clearly continuous. Similarly, the map: $h \to y\hat{\delta}_h(x)$ is continuous and compactly supported. Lemma 11(iv) and (v) show $\mathscr{D}_H \subset M(A \times_\delta G)$. Hence $\Psi(x) \in M(A \times_\delta G)$. Let ξ, $\eta \in \mathscr{H} \otimes L^2(G)$, and let $\omega_{\xi,\eta}$ be the linear functional on $B(\mathscr{H} \otimes L^2(G))$ defined by $T \to \langle T(\xi), \eta \rangle_{\mathscr{H} \otimes L^2(G)}$. Then

$$\begin{aligned}&\omega_{\xi,\eta}(\Psi(x)\, y) \\ &= \lim_{i \to \infty} \sum_{j,l} \omega_{\xi,\eta}(\delta(\delta_u(a_{ij}))(1 \otimes (\varphi(f_{ij}) \cdot g_{il}))\, \delta(\delta_v(b_{il}))) \\ &= \lim_{i \to \infty} \sum_{j,l} \omega_{\xi,\eta}(\delta(\delta_u(a_{ij})) \int_H \hat{\delta}_h(1 \otimes f_{ij})\, dh\ (1 \otimes g_{il})\, \delta(\delta_v(b_{il}))) \\ &= \lim_{i \to \infty} \int_H \omega_{\xi,\eta}\left(\hat{\delta}_h\left(\sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij})\right) \sum_{l=1}^{m_i} (1 \otimes g_{il})\, \delta(\delta_v(b_{il}))\right) dh \\ &= \int_H \omega_{\xi,\eta}(\hat{\delta}_h(x)\, y)\, dh\end{aligned}$$

(by the dominated convergence theorem since for i sufficiently large we have that

$$|\omega_{\xi,\eta}(\gamma_i(h))| \leqslant (\|x\|+1)(\|y\|+1)\,\|\xi\|\,\|\eta\|\,\chi_{FE^{-1}}(h))$$
$$= \omega_{\xi,\eta}\left(\int_H \hat{\delta}_h(x)\,y\,dh\right),$$

which establishes the first equation of the lemma. Now, from this and the fact that: $h \to y\hat{\delta}_h(x)$ is integrable, we have

$$y\Psi(x)w = y\int_H \hat{\delta}_h(x)w\,dh = \int_H y\hat{\delta}_h(x)\,dh\,w \qquad \forall w \in \mathscr{D}.$$

Letting w run over the net of Lemma 17(ii) we obtain the second equation. ∎

THEOREM 19. *Let w, $y \in \mathscr{D}$. Then $\langle w, y\rangle_{\mathscr{D}} = \Psi(w^*y)$ defines a pre-$\mathscr{D}_H$-valued inner product on $\mathscr{D}$. $\mathscr{D}$ equipped with this pre-inner product is a $\mathscr{D}_H$-rigged space. If we define a left action of $\mathscr{D}$ on $\mathscr{D}$ by $y \bullet w = yw$, then $\mathscr{D}$ becomes a pre-Hermitian $\mathscr{D}_H$-rigged $\mathscr{D}$-module.*

Proof. Suppose w, $y \in \mathscr{D}$, $z \in \mathscr{D}_H$, and $(x_j)_{j \in I \times \mathscr{E}_1}$ is the net of Lemma 17(ii), with the subgroup being trivial. Then $\langle \cdot,\cdot\rangle_{\mathscr{D}}$ is clearly conjugate linear in the first variable and linear in the second. By Lemma 18 we have that

$$\langle y, y\rangle_{\mathscr{D}} = \Psi(y^*y)$$
$$= \underset{\text{limit}}{{*}\,\text{strong}}\; x_j\Psi(y^*y)x_j^*$$
$$= \underset{\text{limit}}{{*}\,\text{strong}} \int_H x_j\hat{\delta}_h(y^*y)x_j^*\,dh.$$

Also by Lemma 18 the integrand: $h \to x_j\,\hat{\delta}_h(y^*y)x_j^*$ is norm continuous and compactly supported so it can be uniformly approximated in norm by sums of the form

$$\sum_k v_{ijk}x_j\,\hat{\delta}_{h_{ijk}}(y^*)(x_j\,\hat{\delta}_{h_{ijk}}(y^*))^*, \qquad h_{ijk} \in H,\ v_{ijk} \geqslant 0.$$

Hence

$$\langle y, y\rangle_{\mathscr{D}} = \underset{\text{limit}}{{*}\,\text{strong}}\left(\underset{\text{limit}}{\text{norm}} \sum_k v_{ijk}x_j\,\hat{\delta}_{h_{ijk}}(y^*)(x_j\,\hat{\delta}_{h_{ijk}}(y^*))^*\right) \geqslant 0,$$

since the positive elements are $*$-strongly (and norm) closed. Again by Lemma 18

$$\begin{aligned}\langle w, y\rangle_{\mathscr{D}}^* &= \underset{\text{limit}}{{*}\,\text{strong}} \,(\Psi(w^*y)x_j^*)^* \\ &= \underset{\text{limit}}{{*}\,\text{strong}} \left(\int_H \hat{\delta}_h(w^*y)x_j^*\,dh\right)^* \\ &= \underset{\text{limit}}{{*}\,\text{strong}} \, x_j\Psi(y^*w) \\ &= \langle y, w\rangle_{\mathscr{D}},\end{aligned}$$

Thus $\langle\cdot,\cdot\rangle_{\mathscr{D}}$ is a pre-$\mathscr{D}_H$-valued inner product on $\mathscr{D}$, as claimed. To see that $\mathscr{D}$ is a $\mathscr{D}_H$-rigged space we need to show the conditions of [21, Definition 2.8]. Firstly, since $\hat{\delta}_h(z)=z$ we have that

$$\begin{aligned}\langle w, y\bullet z\rangle_{\mathscr{D}} &= \underset{\text{limit}}{{*}\,\text{strong}} \, x_j\Psi(w^*yz) \\ &= \underset{\text{limit}}{{*}\,\text{strong}} \int_H x_j\hat{\delta}_h(w^*y)\,dh\, z \\ &= \underset{\text{limit}}{{*}\,\text{strong}} \, x_j\Psi(w^*y)z \\ &= \langle w, y\rangle_{\mathscr{D}}\, z.\end{aligned}$$

Let $(z_{(i,E)})$ be the net of Lemma 17(ii), with the subgroup being H. Let $y\in\mathscr{D}$. Choose $f_E\in C_c^+(G)$ such that $\varphi(f_E)=\omega_E^2$ and let $z\in\mathscr{D}_H$. Then

$$\begin{aligned}&\langle\delta(\delta_\psi(e_i))(1\otimes f_E^{1/2}), (\delta(\delta_\psi(e_i))(1\otimes f_E^{1/2}))\bullet z\rangle_{\mathscr{D}}\, y \\ &= \langle\delta(\delta_\psi(e_i))(1\otimes f_E^{1/2}), \delta(\delta_\psi(e_i))(1\otimes f_E^{1/2})\rangle_{\mathscr{D}}\, zy \\ &= \Phi(\delta(\delta_\psi(e_i))(1\otimes f_E)\,\delta(\delta_\psi(e_i)))\, zy \\ &= \int_H \hat{\delta}_h(\delta(\delta_\psi(e_i))(1\otimes f_E)\,\delta(\delta_\psi(e_i)))\, zy\, dh \\ &= \delta(\delta_\psi(e_i))(1\otimes\varphi(f_E))\,\delta(\delta_\psi(e_i))\, zy \\ &= \delta(\delta_\psi(e_i))(1\otimes\omega_E^2)\,\delta(\delta_\psi(e_i))\, zy \\ &= z_{(i,E)}z_{(i,E)}^*zy.\end{aligned}$$

Hence $\langle\delta(\delta_\psi(e_i))(1\otimes f_E^{1/2}),\ \delta(\delta_\psi(e_i))(1\otimes f_E^{1/2})z\rangle_{\mathscr{D}} = z_{(i,E)}z_{(i,E)}^*z\to z$ in norm. So the linear span of the inner product is dense in $\mathscr{D}_H$. Summing up,

$\mathscr{D}$ equipped with this pre-inner product is a $\mathscr{D}_H$-rigged space. We will now show that $\mathscr{D}$ is a pre-Hermitian $\mathscr{D}_H$-rigged $\mathscr{D}$-module. The left action is well defined by Lemma 11(iii). Now we must show that this action satisfies the conditions of [21, Definition 4.19]. Let $x,\ y \in \mathscr{D}$ and $z \in \mathscr{D}_H$. Then $(y \bullet x) \bullet z = yxz = y \bullet (x \bullet z)$ and $\langle y \bullet w, x \rangle_{\mathscr{D}} = \Psi(w^* y^* x) = \langle w, y^* \bullet x \rangle_{\mathscr{D}}$. By Lemmas 18 and 17(ii) we have that

$$\begin{aligned}
&\|y\|^2_{A \times_\delta G} \langle w, w \rangle_{\mathscr{D}} - \langle y \bullet w, y \bullet w \rangle_{\mathscr{D}} \\
&\quad = \mathop{*\,\text{strong}}_{\text{limit}} x_j (\|y\|^2_{A \times_\delta G} \langle w, w \rangle_{\mathscr{D}} - \langle y \bullet w, y \bullet w \rangle_{\mathscr{D}}) x_j^* \\
&\quad = \mathop{*\,\text{strong}}_{\text{limit}} \int_H x_j \hat{\delta}_h (\|y\|^2_{A \times_\delta G} w^* w - w^* y^* y w) x_j^* \, dh \\
&\quad \geqslant 0,
\end{aligned} \tag{5}$$

(since the $\|y\|^2_{A \times_\delta G} - y^* y \geqslant 0$ and the positive elements of $B(\mathscr{H} \otimes L^2(G))$ are $*$-strongly closed). But an element is positive in $B(\mathscr{H} \otimes L^2(G))$ if, and only if, it is positive in any C^*-subalgebra of $B(\mathscr{H} \otimes L^2(G))$ containing it [26, I, Theorem 6.1 and Proposition 4.8]. Thus it is positive in the completion, $A \times_\delta (G/H)$, of $\mathscr{D}_H$, as required. Letting H be the trivial subgroup in Lemma 17(ii), we see that there is a net $(z_{(i,E)})$ in $\mathscr{D}$ such that $z_{(i,E)} x \to x$ for all $x \in \mathscr{D}$. This shows that $\mathscr{D}\mathscr{D}$ is dense in $\mathscr{D}$ for the norm of $B(\mathscr{H} \otimes L^2(G))$. To see that $\mathscr{D} \bullet \mathscr{D} = \mathscr{D}\mathscr{D}$ is dense in $\mathscr{D}$ for the semi-norm $\|\cdot\|_{\mathscr{D}}$ we note that if $x \in \mathscr{D}$, then x is (u, F) for some $u \in A_c(G)$ and some compact subset F of G, and that

$$\|x\|^2_{\mathscr{D}} = \|\langle x, x \rangle_{\mathscr{D}}\| = \|\Psi(x^* x)\| \leqslant \zeta^2_{u,F} \|x\|^2,$$

by Lemma 15, where $\zeta_{u,F}$ is the constant given by that lemma. This shows the left action satisfies the necessary conditions and therefore establishes the theorem. ∎

COROLLARY 20. *We note the map:* $x \to [\theta_x]: \mathscr{D} \to L(\mathscr{D})$, *where* $[\theta_x]$ *is the equivalence class of the operator defined by* $\theta_x(y) = x \bullet y$, *for* $y \in \mathscr{D}$, *is a $*$-homomorphism which is norm-decreasing for the* $A \times_\delta G$*-norm and hence extends to a $*$-homomorphism* $j_{A \times_\delta G}: A \times_\delta G \to L(\mathscr{D})$.

Proof. Standard. ∎

Now the module $\mathscr{D}$ of Theorem 19 can be factored and completed with respect to the semi-norm $\|\cdot\|_{\mathscr{D}} = \|\langle \cdot, \cdot \rangle_{\mathscr{D}}\|^{1/2}$ to give a Hermitian $A \times_\delta (G/H)$-rigged $A \times_\delta G$ module X which can be used to construct representations of $A \times_\delta G$ from those of $A \times_{\delta|} (G/H)$.

To recap: suppose ν: $A \times_{\delta|} (G/H) \to B(\mathcal{Q})$ is a representation of $A \times_{\delta|} (G/H)$ on $\mathcal{Q}$ and Γ is the isomorphism of Proposition 7. Then we can define a pre-inner product on the tensor product $X \otimes \mathcal{Q}$ by

$$\langle [x] \otimes \xi, [y] \otimes \eta \rangle_{X \otimes \mathcal{Q}} = \langle \{\nu \circ \Gamma^{-1}(\langle y, x \rangle_{\mathcal{D}})\}(\xi), \eta \rangle_{\mathcal{Q}}.$$

We obtain a Hilbert space $\operatorname{ind}_{A \times_{\delta|}(G/H)}^{A \times_\delta G} \mathcal{Q}$ from $X \otimes \mathcal{Q}$ by factoring out by the vectors of length zero and completing. The representation of $A \times_\delta G$ induced from ν is then the representation $\operatorname{ind}_{A \times_{\delta|}(G/H)}^{A \times_\delta G} \nu$ of $A \times_\delta G$ on $\operatorname{ind}_{A \times_{\delta|}(G/H)}^{A \times_\delta G} \mathcal{Q}$, determined by

$$\{\{\operatorname{ind}_{A \times_{\delta|}(G/H)}^{A \times_\delta G} \nu\}(y)\}([x] \otimes \xi) = [yx] \otimes \xi, \qquad y \in \mathcal{D}, x \in X, \xi \in \mathcal{Q}.$$

Thus we obtain the map

$$\begin{aligned} \operatorname{ind}_{A \times_{\delta|}(G/H)}^{A \times_\delta G} &: \operatorname{Rep}(A \times_{\delta|} (G/H)) \to \operatorname{Rep}(A \times_\delta G) \\ &: \mu \to \operatorname{ind}_{A \times_{\delta|}(G/H)}^{A \times_\delta G} \mu, \end{aligned}$$

which is the desired induction process.

In [7], Gootman and Lazar introduce a notion of induction for crossed products by coactions, namely, they define the representation of $A \times_\delta G$ induced from a representation ϱ of A to be $((\varrho \otimes i) \circ \delta) \times (1 \otimes M_G)$. In the following proposition we show that this notion of induced representation is the special case, when $H = G$, of ours.

Proposition 21. *Suppose ϱ is a representation of A on the Hilbert space $\mathcal{P}$ and that 1 is the trivial representation of $C_0(G/G)$ on $\mathcal{P}$. Then $(\varrho, 1)$ is a covariant representation of $(A, \delta|, G/G)$ on $\operatorname{ind}_{A \times_{\delta|}(G/G)}^{A \times_\delta G} \mathcal{P}$ and $\operatorname{ind}_{A \times_{\delta|}(G/G)}^{A \times_\delta G} (\varrho \times 1)$ is unitarily equivalent to $((\varrho \otimes i) \circ \delta) \times (1 \otimes M_G)$.*

Proof. X-$\operatorname{ind}_{A \times_{\delta|}(G/G)}^{A \times_\delta G} (\varrho \times 1)$ is unitarily equivalent to $\mathcal{D}$-$\operatorname{ind}_{A \times_{\delta|}(G/G)}^{A \times_\delta G} (\varrho \times 1)$ so it will be enough to show that $\mathcal{D}$-$\operatorname{ind}_{A \times_{\delta|}(G/G)}^{A \times_\delta G} (\varrho \times 1)$ is unitarily equivalent to $(\varrho \otimes i) \circ \delta \times (1 \otimes M_G)$. Let E be the subspace of $\mathcal{D}$ consisting of the elements $\sum_{j=1}^n (1 \otimes (u * f_j))\, \delta(\delta_v(a_j))$ for u and v fixed elements of $A_c(G)$, $f_j \in C_c(G)$, and $a_j \in A$. By Theorem 12 and the continuity of the involution, the set of adjoints of elements of $\mathcal{D}$ is dense in $A \times_\delta G$. This, and the fact that $A_c(G)$ contains an approximate identity for $C^*(G)$, show that E is dense in $\mathcal{D}$ (for the $A \times_\delta G$ norm). We will show that this implies $E \otimes \mathcal{P}$ is dense in $\mathcal{D} \otimes \mathcal{P}$, and thus in $\operatorname{ind}_{A \times_{\delta|}(G/G)}^{A \times_\delta G} \mathcal{P}$. Let $x \in E$. Then $x^* x \in \mathcal{D}$ so by assumption there exists a compact set F such that

$$x^* x = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_w(b_{ij}))(1 \otimes g_{ij})$$

for some $w \in A_c(G)$, $b_{ij} \in A$, and $g_{ij} \in C_F(G)$. Let $\xi \in \mathscr{P}$. Then

$$\begin{aligned}\|x \otimes \xi\|^2_{\mathscr{D} \otimes \mathscr{P}} &= |\langle x \otimes \xi, x \otimes \xi \rangle_{\mathscr{D} \otimes \mathscr{P}}| \\ &= |\langle \{(\varrho \times 1) \circ \Gamma^{-1}(\langle x, x \rangle_{\mathscr{D}})\}(\xi), \xi \rangle_{\mathscr{P}}|\end{aligned}$$

(where Γ is the map of Proposition 7)

$$\begin{aligned}&\leqslant \|\Gamma^{-1} \circ \Psi(x^* x)\|_{A \times_{\delta|} (G/G)} \|\xi\|^2 \\ &= \left\| \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta| (\delta_w(b_{ij}))(1 \otimes \varphi(g_{ij})) \right\|_{A \times_{\delta|} (G/G)} \|\xi\|^2 \\ &\leqslant \zeta_{w,F} \left\| \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_w(b_{ij}))(1 \otimes g_{ij}) \right\|_{A \times_\delta G} \|\xi\|^2\end{aligned}$$

(by Lemma 15, where $\zeta_{w,F}$ is the constant of that lemma)

$$= \zeta_{w,F} \|x\|^2_{A \times_\delta G} \|\xi\|^2.$$

Hence $E \otimes \mathscr{P}$ is dense in $\mathscr{D} \otimes \mathscr{P}$. Now suppose that $\sum_{i=1}^n (1 \otimes f_i)\, \delta(a_i)$, $\sum_{j=1}^m (1 \otimes g_j)\, \delta(b_j) \in E$ and $\xi, \eta \in \mathscr{P}$. Then

$$\begin{aligned}&\left\langle \sum_{i=1}^n ((1 \otimes f_i)\, \delta(a_i)) \otimes \xi, \sum_{j=1}^m ((1 \otimes g_j)\, \delta(b_j)) \otimes \eta \right\rangle_{\mathscr{D} \otimes \mathscr{P}} \\ &= \left\langle \left\{ (\varrho \times 1) \circ \Gamma^{-1} \left(\left\langle \sum_{j=1}^m (1 \otimes g_j)\, \delta(b_j), \sum_{i=1}^n (1 \otimes f_i)\, \delta(a_i) \right\rangle_{\mathscr{D}} \right) \right\} (\xi), \eta \right\rangle_{\mathscr{P}} \\ &= \sum_{i,j} \langle \{ \{\varrho \times 1\}(\Gamma^{-1}(\delta(b_j^*)(1 \otimes \varphi(\bar{g}_j \cdot f_i))\, \delta(a_i)))\}(\xi), \eta \rangle_{\mathscr{P}}\end{aligned}$$

(since $\Psi(\delta(a)(1 \otimes f)\, \delta(b))\, y = \int_H \delta(a)(1 \otimes f^h)\, \delta(b)\, y\, dh = \delta(a)(1 \otimes \varphi(f))$ $\delta(b) y$ for all $y \in \mathscr{D}$, see Lemma 18)

$$= \sum_{i,j} \int_G \overline{g_j(s)}\, f_i(s)\, ds \langle \{ \{\varrho \times 1\}(\delta| (b_j^* a_i))\}(\xi), \eta \rangle_{\mathscr{P}}$$

(since $\varphi(\bar{g}_j \cdot f_i)$ is the constant function with value $\int_G \overline{g_j(s)} f_i(s)\, ds$)

$$\begin{aligned}&= \sum_{i,j} \int_G \overline{g_j(s)}\, f_i(s)\, ds \langle \{\varrho(b_j^* a_i)\}(\xi), \eta \rangle_{\mathscr{P}} \\ &= \sum_{i,j} \int_G \langle \{ \{\varrho(a_i)\}(\xi) \otimes f_i\}(s), \{ \{\varrho(b_j)\}(\eta) \otimes g_j\}(s) \rangle_{\mathscr{P}}\, ds \qquad (6) \\ &= \left\langle \sum_{i=1}^n \{\varrho(a_i)\}(\xi) \otimes f_i, \sum_{j=1}^m \{\varrho(b_j)\}(\eta) \otimes g_j \right\rangle_{\mathscr{P} \otimes L^2(G)}.\end{aligned}$$

Hence $\|\sum_{i=1}^n (1\otimes f_i)\,\delta(a_i)\otimes\xi\|_{\mathscr{D}\otimes\mathscr{P}} = \|\sum_{i=1}^n \{\varrho(a_i)\}(\xi)\otimes f_i\|_{\mathscr{P}\otimes L^2(G)}$ and the map $V\colon E\otimes\mathscr{P}\to\mathscr{P}\otimes L^2(G)$ defined by

$$\sum_{i=1}^n ((1\otimes f_i)\,\delta(a_i))\otimes\xi \to \sum_{i=1}^n \{\varrho(a_i)\}(\xi)\otimes f_i$$

is well defined. We shall show V extends to a unitary operator from $\operatorname{ind}_{A\times_{\delta|}(G/G)}^{A\times_\delta G}\mathscr{P}$ onto $\mathscr{P}\otimes L^2(G)$, which intertwines the actions of $\mathscr{D}$-$\operatorname{ind}_{A\times_{\delta|}(G/G)}^{A\times_\delta G}\varrho\times 1$ and $((\varrho\times 1)\circ\delta)\times(1\otimes M_G)$. It is clear from (6) that V preserves the inner products. Since $\mathscr{A}=\{\delta_u(a)\colon a\in A,\ u\in A_c(G)\}$ is dense in A and since ϱ is a (non-degenerate) representation, $\{\varrho(\mathscr{A})\}(\mathscr{P})\otimes C_c(G)$ is dense in $\mathscr{P}\otimes L^2(G)$. Hence V maps onto a dense subspace of $\mathscr{P}\otimes L^2(G)$. To see that V intertwines the actions, suppose $\eta\otimes h\in\mathscr{P}\otimes L^2(G)$, $(1\otimes f)\,\delta(a)\in\mathscr{D}$, and $z=\sum_{i=1}^n ((1\otimes(\check{u}*g_i))\,\delta(b_i))\otimes\xi\in E\otimes\mathscr{P}$. Then

$$\langle V(\{\operatorname{ind}_{A\times_{\delta|}(G/G)}^{A\times_\delta G}((1\otimes f)\,\delta(a))\}(z)),\eta\otimes h\rangle_{\mathscr{P}\otimes L^2(G)}$$

$$=\sum_{i=1}^n \langle V(((1\otimes f)\,\delta(a)(1\otimes(\check{u}*g_i))\,\delta(b_i))\otimes\xi),\eta\otimes h\rangle_{\mathscr{P}\otimes L^2(G)}$$

$$=\sum_{i=1}^n \left\langle V(((1\otimes f)\int_G (1\otimes(g_i)_s)\,\delta(\delta_{u_s}(a))\,ds\,\delta(b_i))\otimes\xi),\eta\otimes h\right\rangle_{\mathscr{P}\otimes L^2(G)}$$

(by Proposition 8)

$$=\sum_{i=1}^n \int_G \langle\{\varrho(\delta_{u_s}(a)b_i)\}(\xi)\otimes(f\cdot(g_i)_s),\eta\otimes h\rangle_{\mathscr{P}\otimes L^2(G)}\,ds$$

$$=\sum_{i=1}^n \int_G\int_G f(t)\,g_i(s^{-1}t)\,\overline{h(t)}\,\langle\{\varrho(\delta_{u_s}(a)b_i)\}(\xi),\eta\rangle_{\mathscr{P}}\,dt\,ds.$$

Writing $\delta(a)$ as the strict limit of the net $(\sum_{j=1}^{n_k} a_{jk}\otimes\lambda_G(\gamma_{jk}))$ for some $a_{jk}\in A$ and $\gamma_{jk}\in C_c(G)$, so that $\delta_{u_s}(a)$ is the limit of $(\sum_{j=1}^{n_k}\int_G u(s^{-1}r)\,\gamma_{jk}(r)\,dr\;a_{jk})$, we obtain that the above equals

$$=\int_G\int_G \lim_{k\to\infty}\sum_{i,j}\int_G u(s^{-1}r)\,\gamma_{jk}(r)\,dr\,f(t)$$
$$\times g_i(s^{-1}t)\,\overline{h(t)}\,dt\langle\{\varrho(a_{jk}b_i)\}(\xi),\eta\rangle_{\mathscr{P}}\,ds$$

$$=\lim_{k\to\infty}\sum_{i,j}\int_G\int_G\int_G \gamma_{jk}(r)\,u(s^{-1}r)\,g_i(s^{-1}t)$$
$$\times f(t)\,\overline{h(t)}\,dr\,dt\,ds\langle\{\varrho(a_{jk}b_i)\}(\xi),\eta\rangle_{\mathscr{P}}$$

(by the dominated convergence theorem, since the strong convergence of the net $(\sum_j \int_G u(s^{-1}r)\,\gamma_{jk}(r)\,dr\,a_{jk}b_i)$ to $\delta_{u_s}(a)b_i$ implies that for i sufficiently large

$$\left|\sum_{i=1}^{n} f(t)\,g_i(s^{-1}t)\,\overline{h(t)}\left\langle\left\{\varrho\left(\sum_{j=1}^{n_k}\int_G u(s^{-1}r)\,\gamma_{jk}(r)\,dr\,a_{jk}b_i\right)\right\}(\xi),\eta\right\rangle_{\mathscr{P}}\right|$$

$$\leqslant \sum_{i=1}^{n} \|g_i\|_{C_0(G)}\,|f(t)\,\overline{h(t)}|\,(\|u\|_{A(G)}\,\|a\|_A\,\|b_i\|_A\,\|\xi\|+1)\,\|\eta\|\,\chi_F(s),$$

where F is the compact set $\bigcup_{i=1}^{n}(\operatorname{supp} f)(\operatorname{supp} g_i)^{-1})$

$$= \lim_{k\to\infty}\sum_{i,j}\int_G \{\gamma_{jk} * \check{u} * g_i\}(t)\,f(t)\,\overline{h(t)}\,dt\langle\{\varrho(a_{jk}b_i)\}(\xi),\eta\rangle_{\mathscr{P}}$$

$$= \lim_{k\to\infty}\sum_{i,j}\langle\{\varrho(a_{jk}b_i)\}(\xi)\otimes((\gamma_{jk} * \check{u} * g_i)\cdot f),\eta\otimes h\rangle_{\mathscr{P}\otimes L^2(G)}$$

$$= \lim_{k\to\infty}\sum_{i,j}\langle\{\varrho(a_{jk})\otimes(M_G(f)\,\lambda_G(\gamma_{jk}))\}$$
$$(\{\varrho(b_i)\}(\xi)\otimes(\check{u} * g_i)),\eta\otimes h\rangle_{\mathscr{P}\otimes L^2(G)}$$

$$= \sum_{i=1}^{n}\langle\{(1\otimes M_G(f))\{\varrho\otimes i\}(\delta(a))\}$$
$$(\{\varrho(b_i)\}(\xi)\otimes(\check{u} * g_i)),\eta\otimes h\rangle_{\mathscr{P}\otimes L^2(G)}$$

$$= \left\langle\left\{\{((\varrho\otimes i)\circ\delta)\times(1\otimes M_G)\}((1\otimes f)\,\delta(a))\right\}\right.$$
$$\left.\left(V\left(\sum_{i=1}^{n}((1\otimes(\check{u} * g_i))\,\delta(b_i))\otimes\xi\right)\right),\eta\otimes h\right\rangle.$$

This establishes the proposition. ∎

5. The Imprimitivity Theorem

In this section we determine the imprimitivity algebra for the $\mathscr{D}_H$-rigged space $\mathscr{D}$ (in terms of $A\times_\delta G$ and other known quantities). This enables us to interpret Rieffel's imprimitivity theorem [21, Theorem 6.29] and obtain criteria that allows the characterization of those representations of $A\times_\delta G$ which can be constructed from representations of $A\times_{\delta|}(G/H)$ by the induction process of Section 4. That is, we present an imprimitivity theorem for the process. Throughout this section H will be a closed normal amenable subgroup of G and δ will be a non-degenerate coaction of G on A.

PROPOSITION 22. *Suppose $\xi \in \mathscr{I}_H$ and $x \in \mathscr{D}$. If we define a left action of $\mathscr{I}_H$ on $\mathscr{D}$ by $\xi \bullet x = \int_H \sqrt{\Delta h}\, \xi(h)\, \hat{\delta}_h(x)\, dh$, then $\mathscr{D}$ becomes a possibly degenerate (that is, $\mathscr{I}_H \bullet \mathscr{D}$ need not be dense in $\mathscr{D}$) pre-Hermitian $\mathscr{I}_H$-rigged $\mathscr{D}_H$ bimodule.*

Proof. Firstly we show that $\xi \bullet x$ does in fact belong to $\mathscr{D}$. By assumption there exist compact subsets E, F of G and $u, v \in A_c(G)$ such that x is (v, F) and $\xi(h)$ is (u, E) for all $h \in H$. By Lemma 11(vi) there exists a compact subset D of G such that $\hat{\delta}_h(x)$ is (v, D) for all $h \in \operatorname{supp} \xi$. By Lemma 11(iii) there exists $\omega \in A_c(G)$ such that $\xi(h)\, \hat{\delta}_h(x)$ is (ω, D) for all $h \in \operatorname{supp} \xi$ and hence for all $h \in H$. It is clear that: $h \to \sqrt{\Delta h}\, \xi(h)\, \hat{\delta}_h(x)$ is also continuous and compactly supported, hence an element of $\mathscr{I}_H$. Thus by a familiar approximation argument (cf. the proof of Proposition 13(iii)) the integral $\xi \bullet x$ can be written as the limit of elements which are (ω, D) and is thus (ω, D) itself. Hence $\xi \bullet x \in \mathscr{D}$. It is now routine to show that $\mathscr{D}$ is an $\mathscr{I}_H$-$\mathscr{D}_H$ bimodule. Let $w, x, y \in \mathscr{D}$ and $\xi \in \mathscr{I}_H$. Then by Lemma 18

$$\begin{aligned}
\langle \xi \bullet x, y \rangle_{\mathscr{D}}\, w &= \int_H \hat{\delta}_h((\xi \bullet x)^* y) w\, dh \\
&= \int_H \int_H \sqrt{\Delta r}\, \hat{\delta}_h(\hat{\delta}_r(x^*)\, \xi(r)^* y) w\, dr\, dh \\
&= \int_H \int_H \frac{1}{\sqrt{\Delta r}}\, \hat{\delta}_{hr}\{\hat{\delta}_{r^{-1}}(x^*)\, \xi(r^{-1})^* y\} w\, dr\, dh \\
&= \int_H \hat{\delta}_h\left(x^* \left(\int_H \sqrt{\Delta r} \left(\frac{1}{\Delta r} \hat{\delta}_r(\xi(r^{-1})^*)\right) \hat{\delta}_r(y)\, dr \right)\right) w\, dh \\
&= \langle x, \xi^* \bullet y \rangle_{\mathscr{D}}\, w.
\end{aligned}$$

Letting w run over the net of Lemma 17(ii) shows $\langle \xi \bullet x, y \rangle_{\mathscr{D}} = \langle x, \xi^* \bullet y \rangle_{\mathscr{D}}$. Define an action of H on $\mathscr{D}$ by $h \bullet x = \sqrt{\Delta h}\, \hat{\delta}_h(x)$, for $h \in H$. By Lemma 18

$$\begin{aligned}
w\langle h \bullet x, h \bullet x \rangle_{\mathscr{D}} &= \int_H \Delta h\, w\, \hat{\delta}_{rh}(x^* x)\, dr \\
&= \int_H w\, \hat{\delta}_r(x^* x)\, dr = w\langle x, x \rangle_{\mathscr{D}}.
\end{aligned}$$

Hence $\langle h \bullet x, h \bullet x \rangle_{\mathscr{D}} = \langle x, x \rangle_{\mathscr{D}}$. Let p be a state on $A \times_\delta (G/H)$. Then $p(\langle \cdot, \cdot \rangle_{\mathscr{D}})$ is a scalar valued pre-inner product on $\mathscr{D}$. Suppose $\mathscr{D}_p$ is the Hilbert space obtained by factoring $\mathscr{D}$ by vectors of length zero and completing and that $\|\cdot\|_p$ is the resulting norm on $\mathscr{D}_p$. Then by (5) and the above, $\|yx\|_p \leqslant \|y\|_{A \times_\delta G} \|x\|_p$ and $\|\sqrt{\Delta h}\, \hat{\delta}_h(x)\|_p = \|x\|_p$. Thus

$$\begin{aligned}
p(\langle \xi \bullet x, \xi \bullet x \rangle_{\mathscr{D}})^{1/2} &= \left\| \int_H \sqrt{\Delta h}\, \xi(h)\, \hat{\delta}_h(x)\, dh \right\|_p \\
&\leqslant \int_H \|\xi(h) \sqrt{\Delta h}\, \hat{\delta}_h(x)\|_p\, dh \\
&\leqslant \int_H \|\xi(h)\|_{A \times_\delta G}\, dh\, \|x\|_p \\
&\leqslant \|\xi\|_1\, p(\langle x, x \rangle_{\mathscr{D}})^{1/2},
\end{aligned}$$

for every state on $A \times_\delta (G/H)$. Hence $\langle \xi \bullet x, \xi \bullet x \rangle_{\mathscr{D}} \leqslant \|\xi\|_1^2 \langle x, x \rangle_{\mathscr{D}}$ and $\mathscr{D}$ is a possibly degenerate pre-Hermitian $\mathscr{D}_H$-rigged $\mathscr{I}_H$ module, as claimed. ∎

COROLLARY 23. *The map : $\xi \to [\theta_\xi]$: $\mathscr{I}_H \to L(\mathscr{D})$, where $[\theta_\xi]$ is the equivalence class of the operator defined by $\theta_\xi(y) = \xi \bullet x$, for all $x \in \mathscr{D}$, is a $*$-homomorphism which is norm-decreasing for the C^*-norm on $(A \times_\delta G) \times_{\hat{\delta}} H$ and hence extends to a $*$-homomorphism Θ: $(A \times_\delta G) \times_{\hat{\delta}} H \to L(\mathscr{D})$.*

Proof. The map : $\xi \to [\theta_\xi]$ is easily seen to be a $*$-homomorphism which is norm-decreasing for the L^1-norm on $L^1(H, A \times_\delta G)$. Hence, by the universality property of the enveloping C^*-algebra, it is also norm-decreasing for the C^*-norm. ∎

We now wish to show the linear span $\mathscr{O}$ of the maps $\gamma_{x,y}(h) = (1/\sqrt{\Delta h})\, x\hat{\delta}_h(y^*)$, for $x, y \in \mathscr{D}$, is dense in $(A \times_\delta G) \times_{\hat{\delta}} H$. From this we will be able to conclude that Θ is onto. To do this we need to establish the following lemma. Recall that a net (f_k) in $C_c(H, A \times_\delta G)$ converges to f in the inductive limit topology if, and only if, the f_k converge to f uniformly (in the norm of $A \times_\delta G$) and there exists a compact subset F of G such that $\operatorname{supp} f_k \subset F$ for all k sufficiently large.

LEMMA 24. *$\mathscr{O}$ contains a net $(z_\alpha)_{\alpha \in \mathscr{A}}$ such that if $\xi \in \mathscr{I}_H$ is of the form $\xi(h) = \eta(h)(1 \otimes (\check{u} * f))\, \delta(a)$, where $\eta \in C_c(H)$, $f \in C_c(G)$, $u \in A_c(G)$, and $a = \delta_v(b)$, for some $b \in A$ and $v \in A_c(G)$, then $z_\alpha * \xi \to \xi$ in the inductive limit topology.*

Proof. The above mentioned net will be indexed by quadruples (N, C, ε, i) where N runs over the relatively compact neighbourhoods of the identity in H which are contained in some fixed compact neighbourhood of the identity, E, C runs over the compact subsets of G, $\varepsilon \in \mathbb{R}^+$, and i runs over the index set of an approximate identity $(e_i)_{i \in I}$ of A. The net is directed by

$$(N, C, \varepsilon, i) \leqslant (N', C', \varepsilon', i') \Leftrightarrow N \supset N', C \subset C', \varepsilon \geqslant \varepsilon', i \leqslant i'.$$

Let (N, C, ε, i) be given. We begin the construction of the net by choosing an open cover $(U_j)_{j=1}^n$ of C such that

$$\{h \in H: U_j h \cap U_j \neq \varnothing\} \subset N. \tag{7}$$

This is possible by the first lemma of [23]. Now we choose $f_j \in C_c^+(U_j)$ such that

$$\sum_{j=1}^{n} f_j \equiv 1 \qquad \text{on} \quad C \tag{8}$$

and is between zero and one elsewhere. Choose $g_j \in C_c^+(U_j)$ such that

$$\left| f_j(t) - g_j(t) \int_H \frac{1}{\sqrt{\Delta h}}\, g_j(th)\, dh \right| < \varepsilon/n \qquad \forall t \in G.$$

This can be done by the second lemma of [23]. We have that

$$\left\| \sum_{j=1}^{n} f_j - \sum_{j=1}^{n} g_j \cdot \int_H \frac{1}{\sqrt{\Delta h}}\, g_j^h\, dh \right\|_{C_0(G)} < \varepsilon. \tag{9}$$

Let $\psi \in A_c(G)$ be positive as an element of $C_r^*(G)^*$, with $\psi(1) = \|\psi\|_{B(G)} = 1$. Then by Lemma 17, $(\delta_\psi(e_i))_{i \in I}$ is an approximate identity of A. Let $d_i = \delta_\psi(e_i)$. Then the required net is given by

$$\begin{aligned} z_\alpha(h) &= \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} \delta(d_i)(1 \otimes g_j)\, \hat{\delta}_h(\delta(d_i)(1 \otimes g_j))^* \\ &= \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} \delta(d_i)(1 \otimes (g_j \cdot g_j^h))\, \delta(d_i). \end{aligned}$$

To show that $z_\alpha * \xi \to \xi$ in the inductive limit topology we will show that for all $\sigma > 0$ there exists $\alpha_0 \in \mathscr{A}$ such that each of the three quantities (i), (ii), and (iii) on the right hand side of the following inequality is less than $\sigma/3$, for all $r \in F = E \operatorname{supp} \xi$ (note that $\operatorname{supp} z_\alpha \subset N \subset E$ by (7), so $\operatorname{supp}(z_\alpha * \xi) \subset F$):

$$\begin{aligned}
&\|(z_\alpha * \xi)(r) - \xi(r)\| \\
&\quad \leqslant \|\delta(d_i^2)\, \xi(r) - \xi(r)\| \\
&\qquad + \left\| (z_\alpha * \xi)(r) - \delta(d_i) \int_G \left(1 \otimes \left(\sum_{j=1}^n \left(\int_H \frac{1}{\sqrt{\Delta h}}\, g_j \cdot g_j^h \, dh \right) \cdot f_s \eta(r) \right)\right) \right. \\
&\qquad\qquad \left. \delta(\delta_{u_s}(d_i))\, ds\, \delta(a) \right\| \\
&\qquad + \left\| \delta(d_i) \int_G \left(1 \otimes \left(\sum_{j=1}^n \left(\int_H \frac{1}{\sqrt{\Delta h}}\, g_j \cdot g_j^h \, dh \right) \cdot f_s \eta(r) \right)\right) \right. \\
&\qquad\qquad \left. \delta(\delta_{u_s}(d_i))\, ds\, \delta(a) - \delta(d_i^2)\, \xi(r) \right\| \\
&\quad = \text{(i)} + \text{(ii)} + \text{(iii)}.
\end{aligned}$$

Let $\sigma > 0$. Choose N_0 such that

$$\|\eta(h^{-1}r) f_s^h - \eta(r) f_s\|_{C_0(G)} < \sigma/(6\kappa\, \|u\|_{B(G)}\, \|a\|_A), \tag{10}$$

for all $h \in N_0$, $r \in H$, and $s \in G$, where $\kappa = \mu_G((\operatorname{supp} \psi)(\operatorname{supp} u)^{-1})$. Choose

C_0 such that $C_0 \supset (\operatorname{supp} \psi)(\operatorname{supp} u)^{-1} (\operatorname{supp} f)$. (11)

Choose

$$\varepsilon_0 = \min(1; \sigma/(3\kappa\, \|\eta\|_{C_0(G)}\, \|f\|_{C_0(G)}\, \|u\|_{B(G)}\, \|a\|_A)). \tag{12}$$

Now $(d_i)_{i \in I}$ is an approximate identity of A. Hence so is $(d_i^2)_{i \in I}$. Since δ is a non-degenerate $*$-homomorphism $\delta(d_i^2) \to 1$ strictly in $M(A \times_\delta G)$. This and the fact that F is compact enable us to choose $i_0 \in I$ such that $i \geqslant i_0$ implies

$$\|\delta(d_i^2)\, \xi(r) - \xi(r)\| \leqslant \sigma/3, \qquad \forall r \in F,$$

that is, (iii) $< \sigma/3$ for all $r \in F$. Now we will show that for $\alpha \geqslant \alpha_0$, (i) $< \sigma/3$, for all $r \in F$. First, note that

$$\begin{aligned}
&(z_\alpha * \xi)(r) \\
&\quad = \int_H \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^n \delta(d_i)(1 \otimes (g_j \cdot g_j^h))\, \delta(d_i)\, \hat{\delta}_h(\xi(h^{-1}r))\, dh
\end{aligned}$$

$$=\int_H \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} \delta(d_i)(1\otimes(g_j\cdot g_j^h))\,\delta(d_i)(1\otimes(\check{u}*f)^h)\,\delta(a)\,\eta(h^{-1}r)\,dh$$

$$=\int_H \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} \delta(d_i)(1\otimes(g_j\cdot g_j^h))\,\delta(d_i)(1\otimes(\check{u}*f^h))\,\delta(a)\,\eta(h^{-1}r)\,dh$$

$$=\int_H \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} \delta(d_i)(1\otimes(g_j\cdot g_j^h))\int_G (1\otimes f_s^h)\,\delta(\delta_{u_s}(d_i))\,ds\,\delta(a)\,\eta(h^{-1}r)\,dh$$

(by Proposition 8, with H the trivial subgroup)

$$=\int_G \delta(d_i)\int_H\left(1\otimes\left(\frac{1}{\sqrt{\Delta h}}\sum_{j=1}^{n} g_j\cdot g_j^h\cdot f_s^h\,\eta(h^{-1}r)\right)\right)dh\,\delta(\delta_{u_s}(d_i))\,\delta(a)\,ds$$

(since the integrand is continuous and compactly supported in $E\times(\operatorname{supp}\psi)(\operatorname{supp}u)^{-1}\subset H\times G$)

$$=\int_G \delta(d_i)\left(1\otimes\left(\sum_{j=1}^{n}\int_H \frac{1}{\sqrt{\Delta h}} g_j\cdot g_j^h\cdot f_s^h\,\eta(h^{-1}r)\,dh\right)\right)\delta(\delta_{u_s}(d_i))\,\delta(a)\,ds$$

(where the inner integral is the integral of $h\to(1/\sqrt{\Delta h})\,\eta(h^{-1}r)\,g_j\cdot g_j^h\cdot f_s^h$ $:H\to C_0(G)\subset B(L^2(G))$). So for $\alpha\geqslant\alpha_0$ we have that

$$\left\|(z_\alpha*\xi)(r)-\delta(d_i)\int_G\left(1\otimes\left(\sum_{j=1}^{n}\left(\int_H\frac{1}{\sqrt{\Delta h}}g_j\cdot g_j^h\,dh\right)\right.\right.\right.$$
$$\left.\left.\left.\cdot f_s\eta(r)\right)\right)\delta(\delta_{u_s}(d_i))\,ds\,\delta(a)\right\|$$

$$\leqslant\int_G\left\|\delta(d_i)\left(1\otimes\left(\sum_{j=1}^{n}\int_H\frac{1}{\sqrt{\Delta h}}g_j\cdot g_j^h\right.\right.\right.$$
$$\left.\left.\left.\cdot(f_s^h\eta(h^{-1}r)-f_s\eta(r))\,dh\right)\right)\right\|ds\,\|u\|_{B(G)}\,\|a\|_A$$

$$\leqslant\int_G\left\|\int_H\frac{1}{\sqrt{\Delta h}}\sum_{j=1}^{n}g_j\cdot g_j^h\right.$$
$$\left.\cdot(f_s^h\eta(h^{-1}r)-f_s\eta(r))\,dh\right\|_{C_0(G)}ds\,\|u\|_{B(G)}\,\|a\|_A$$

(since $\|1\otimes M_G(\gamma)\|=\|\gamma\|_{C_0(G)}$)

$$<\kappa\,2\sigma/(6\kappa\,\|u\|_{B(G)}\,\|a\|_A)\,\|u\|_{B(G)}\,\|a\|_A=\sigma/3.$$

The last inequality follows from the fact that if $t\in G$, then

$$\left|\int_H \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^n g_j(t)\, g_j(th)(f(s^{-1}th)\,\eta(h^{-1}r) - f(s^{-1}t)\,\eta(r))\,dh\right|$$
$$\leqslant \sum_{j=1}^n g_j(t) \int_H \frac{1}{\sqrt{\Delta h}}\, g_j(th)\,|f(s^{-1}th)\,\eta(h^{-1}r) - f(s^{-1}t)\,\eta(r)|\,dh$$
$$< \sum_{j=1}^n g_j(t) \int_H \frac{1}{\sqrt{\Delta h}}\, g_j(th)\,dh\;\sigma/(6\kappa\,\|u\|_{B(G)}\,\|a\|_A)$$

(by (10), since the h-support of the integrand is contained in N_0)

$$\leqslant\left(1 + \sum_{j=1}^n f_j(t)\right)\sigma/(6\kappa\,\|u\|_{B(G)}\,\|a\|_A) \qquad \text{(by (9) and (12))}$$
$$\leqslant 2\sigma/(6\kappa\,\|u\|_{B(G)}\,\|a\|_A) \qquad \text{(by (8))}.$$

So we have that $\alpha \geqslant \alpha_0$ implies (i) $\leqslant \sigma/3$, for all $r \in F$. Now we show that for $\alpha \geqslant \alpha_0$, (ii) $\leqslant \sigma/3$, for all $r \in F$. Since

$$\delta(d_i^2)\,\xi(r) = \eta(r)\,\delta(d_i^2)(1 \otimes (\check{u} * f))\,\delta(a)$$
$$= \eta(r)\,\delta(d_i) \int_G (1 \otimes f_s)\,\delta(\delta_{u_s}(d_i))\,ds\,\delta(a)$$

we have that for all $\alpha \geqslant \alpha_0$

$$\left\|\delta(d_i) \int_G \left(1 \otimes \left(\sum_{j=1}^n \left(\int_H \frac{1}{\sqrt{\Delta h}}\, g_j \cdot g_j^h\,dh\right)\right.\right.\right.$$
$$\left.\left.\left. \cdot f_s \eta(r)\right)\right)\delta(\delta_{u_s}(d_i))\,ds\,\delta(a) - \delta(d_i^2)\,\xi(r)\right\|$$
$$= \left\|\eta(r)\,\delta(d_i) \int_G \left(1 \otimes \left(\left(\sum_{j=1}^n \int_H \frac{1}{\sqrt{\Delta h}}\, g_j \cdot g_j^h\,dh - \sum_{j=1}^n f_j\right)\right.\right.\right.$$
$$\left.\left.\left. \cdot f_s\right)\right)\delta(\delta_{u_s}(d_i))\,ds\,\delta(a)\right\|$$

(since $\sum f_j$ is identically one on C_0 which by (11) contains supp f_s for all s in the s-support of the integrand)

$$\leqslant \|\eta\|_{C_0(G)} \int_G \left\|\sum_{j=1}^n \int_H \frac{1}{\sqrt{\Delta h}}\, g_j \cdot g_j^h\,dh - \sum_{j=1}^n f_j\right\|_{C_0(G)}$$
$$\times \|f_s\|_{C_0(G)}\,\|u_s\|_{B(G)}\,ds\,\|a\|_A$$
$$< \frac{\kappa\sigma\,\|\eta\|_{C_0(G)}\,\|f\|_{C_0(G)}\,\|u\|_{B(G)}\,\|a\|_A}{3\kappa\,\|\eta\|_{C_0(G)}\,\|f\|_{C_0(G)}\,\|u\|_{B(G)}\,\|a\|_A}$$
$$= \sigma/3 \qquad \text{by (9) and (12))}.$$

So we have that $\alpha \geqslant \alpha_0$ implies (ii) $\leqslant \sigma/3$ for all $r \in F$, showing that $z_\alpha * \xi \to \xi$ uniformly on F. This establishes the lemma since the support of the $z_\alpha * \xi$ is contained in F for all α. ∎

LEMMA 25. *The linear span $\mathcal{O}$ of the maps*

$$\gamma_{x,y}(h) = \frac{1}{\sqrt{\Delta h}}\, x\, \hat{\delta}_h(y^*), \qquad x, y \in \mathcal{D}$$

is dense for the inductive limit topology on $C_c(H, A \times_\delta G)$, and hence is dense in $L^1(H, A \times_\delta G)$.

Proof. Firstly, we observe that if $\xi \in \mathcal{I}_H$, then

$$\begin{aligned}
(\gamma_{x,y} * \xi)(h) &= \int_H \frac{1}{\sqrt{\Delta r}}\, x\hat{\delta}_r(y^*\xi(r^{-1}h))\, dr \\
&= \int_H \frac{1}{\sqrt{\Delta(hr)}}\, x\hat{\delta}_{hr}(y^*\xi(r^{-1}))\, dr \\
&= \frac{1}{\sqrt{\Delta h}}\, x\hat{\delta}_h\left(\left(\int_H \sqrt{\Delta r}\left(\frac{1}{\Delta r}\,\hat{\delta}_r(\xi(r^{-1})^*)\right)\hat{\delta}_r(y)\right)^*\right) dr \\
&= \frac{1}{\sqrt{\Delta h}}\, x\hat{\delta}_h((\xi^* \bullet y)^*) \\
&= \gamma_{x,\xi^* \bullet y}(h).
\end{aligned}$$

Let z_α, ψ, and the g_i be as in Lemma 24. Let $x_\alpha^i = \delta(\delta_\psi(e_i))(1 \otimes g_i)$. Then

$$z_\alpha * \xi = \sum_{i=1}^{n} \gamma_{x_\alpha^i, x_\alpha^i} * \xi = \sum_{i=1}^{n} \gamma_{x_\alpha^i, \xi^* \bullet x_\alpha^i} \in \mathcal{O}.$$

Let ξ be of the form $\xi(r) = \eta(r)(1 \otimes (\check{u} * f))\,\delta(a)$, where $\eta \in C_c(H)$, $f \in C_c(G)$, $u \in A_c(G)$, and $a = \delta_v(b)$, for some $b \in A$ and $v \in A_c(G)$. Then by Lemma 24 we have that $\sum_{i=1}^{n} \gamma_{x_\alpha^i, \xi^* \bullet x_\alpha^i} \to \xi$ in the inductive limit topology. Since the linear span of the set of elements ξ of the above special form is dense for the inductive limit topology on $C_c(H, A \times_\delta G)$, the lemma follows. ∎

PROPOSITION 26. *The map Θ: $(A \times_\delta G) \times_{\hat{\delta}} H \to L(\mathcal{D})$ of Corollary 23 is an injective $*$-homomorphism of $(A \times_\delta G) \times_{\hat{\delta}} H$ onto $K(\mathcal{D})$.*

Proof. With the notation of Corollary 23 we have $\Theta(\gamma_{x,y}) = [\theta_{\gamma_{x,y}}]$, where

$$\theta_{\gamma_{x,y}}(z) = \int_H x\hat{\delta}_r(y^*z)\,dr = x \bullet \langle y, z \rangle_{\mathscr{D}} \qquad \forall z \in \mathscr{D},$$

that is, $\Theta(\gamma_{x,y})$ is the generator $[T_{x,y}]$ (see Section 1) of $K(\mathscr{D})$. By Lemma 25 the linear span of the $\gamma_{x,y}$ is dense in $L^1(H, A \times_\delta G)$, and thus in $(A \times_\delta G) \times_{\hat{\delta}} H$. Also the linear span of the $[T_{x,y}]$ is dense in $K(\mathscr{D})$. Hence Θ maps $(A \times_\delta G) \times_{\hat{\delta}} H$ into, and onto, $K(\mathscr{D})$.

Now we show that Θ is injective. By [21, Propositions 6.5 and 6.6] $\mathscr{D}$ is a pre-Hermitian $\mathscr{D}_H$-rigged $K(\mathscr{D})$ module which can be used to induce representations of $A \times_\delta (G/H)$ to representations of $K(\mathscr{D})$. Let Λ be the representation of $K(\mathscr{D})$ on Z induced from i, where i is the representation of $A \times_\delta (G/H)$ (and $A \times_\delta G$) on $\mathscr{H} \otimes L^2(G)$ defined by $i(z) = z$, and Z is $\mathscr{D} \otimes \mathscr{H} \otimes L^2(G)$ factored and completed with respect to the pre-inner product

$$\langle x \otimes \xi, y \otimes \eta \rangle_{\mathscr{D} \otimes \mathscr{H} \otimes L^2(G)} = \langle \{\langle y, x \rangle_{\mathscr{D}}\}(\xi), \eta \rangle_{\mathscr{H} \otimes L^2(G)}.$$

Let $\check{\imath}$ be the representation of $A \times_\delta G$ on $L^2(H, \mathscr{H} \otimes L^2(G))$ determined by $\{\{\check{\imath}(x)\}(\zeta)\}(h) = \{\hat{\delta}_h(x)\}(\zeta(h))$, for $h \in H$ and $\zeta \in C_c(H, \mathscr{H} \otimes L^2(G))$. Let $1 \otimes \rho_H$ be the unitary representation of H on $L^2(H, \mathscr{H} \otimes L^2(G))$ defined by $\{\{1 \otimes \rho_H(r)\}(\zeta)\}(h) = \sqrt{\Delta r}\,\zeta(hr)$. Now $(\check{\imath}, 1 \otimes \rho_H)$ is a covariant representation (the right regular representation) of $(A \times_\delta G, H, \hat{\delta})$. Since H is amenable we have that $(\check{\imath}, 1 \otimes \rho_H)$ is faithful [19, Sect. 7.7.5]. We will show that $\check{\imath} \times (1 \otimes \rho_H)$ is unitarily equivalent to $\Lambda \circ \Theta$. Hence $\Lambda \circ \Theta$ is also faithful and Θ must be injective. Let $\mathscr{F} = \{w(\xi)\colon w \in \mathscr{D},\ \xi \in \mathscr{H} \otimes L^2(G)\}$. By Lemma 17(ii), $\mathscr{D}$ contains a net which converges to 1 strongly in $B(\mathscr{H} \otimes L^2(G))$. Hence $\mathscr{F}$ is dense in $\mathscr{H} \otimes L^2(G)$. Define a map

$$V\colon \mathscr{D} \otimes \mathscr{F} \to L^2(H, \mathscr{H} \otimes L^2(G)) \qquad \text{by} \quad \{V(x \otimes \xi)\}(h) = \{\hat{\delta}_h(x)\}(\xi).$$

Let $\zeta = w(\xi) \in \mathscr{F}$, for $w \in \mathscr{D}$ and $\xi \in \mathscr{H} \otimes L^2(G)$. Then $\{V(x \otimes \zeta)\}(h) = \{\hat{\delta}_h(x)w\}(\xi)$, which shows that $V(x \otimes \xi)$ is continuous and compactly supported and hence in $L^2(H, \mathscr{H} \otimes L^2(G))$ (since, by Lemma 18, the map: $h \to \hat{\delta}_h(x)w$ is continuous and compactly supported).

Let $\varepsilon > 0$. Let g be the element of $C_c(H, \mathscr{F})$ defined by $g(h) = \alpha(h)\,\omega(\xi)$, where $\alpha \in C_c(H)$, $w \in \mathscr{D}$, and $\xi \in \mathscr{H} \otimes L^2(G)$. By Lemma 25, there exists a compact subset F of H and $x_j, y_j \in \mathscr{D}$, for $j = 1, ..., n$, with $\operatorname{supp} \gamma_{y_j^*, x_j} \subset F$ and such that

$$\left\| \sum_{j=1}^n \gamma_{y_j^*, x_j}(h) - \frac{1}{\sqrt{\Delta h}} (\overline{g(h)})^* \right\|^2 < \varepsilon/(\kappa^2 \|\xi\|^2 \mu_H(F)) \qquad \forall h \in F,$$

where κ is the maximum value of Δh on F. Now

$$\begin{aligned}
&\left\| V\left(\sum_{j=1}^{n} x_j \otimes (y_j(\xi)) \right) - g \right\|^2_{L^2(H,\, \mathscr{H} \otimes L^2(G))} \\
&= \int_H \left\| \sum_{j=1}^{n} \{\hat{\delta}_h(x_j)\}(y_j(\xi)) - \alpha(h)\, w(\xi) \right\|^2_{\mathscr{H} \otimes L^2(G)} dh \\
&\leqslant \int_H \left\| \sum_{j=1}^{n} \hat{\delta}_h(x_j)\, y_j - \alpha(h) w \right\|^2_{B(\mathscr{H} \otimes L^2(G))} dh\, \|\xi\|^2 \\
&= \int_H \left\| \sum_{j=1}^{n} y_j^*\, \hat{\delta}_h(x_j^*) - \overline{\alpha(h)}\, w^* \right\|^2 dh\, \|\xi\|^2 \\
&\leqslant \int_H \left\| \sum_{j=1}^{n} \gamma_{y_j^*, x_j}(h) - \frac{1}{\sqrt{\Delta h}} (\overline{g(h)})^* \right\|^2 dh\, \kappa\, \|\xi\|^2 \\
&< \mu_H(F)(\varepsilon/(\kappa^2\, \|\xi\|^2\, \mu_H(F)))\kappa\, \|\xi\|^2 \\
&= \varepsilon.
\end{aligned}$$

Since $C_c(H) \otimes \mathscr{F}$ is dense in $L^2(H, \mathscr{H} \otimes L^2(G))$, V maps onto a dense subspace. Let w, x, $y \in \mathscr{D}$ and ξ, η, $\zeta \in \mathscr{H} \otimes L^2(G)$ with $\zeta = w(\xi)$. Then

$$\begin{aligned}
&\langle x \otimes \zeta,\, y \otimes \eta \rangle_{\mathscr{D} \otimes \mathscr{H} \otimes L^2(G)} \\
&= \langle \{\langle y, x \rangle_{\mathscr{D}}\, w\}(\xi), \eta \rangle_{\mathscr{H} \otimes L^2(G)} \\
&= \left\langle \left\{ \int_H \hat{\delta}_h(y^* x) w\, dh \right\} (\xi), \eta \right\rangle_{\mathscr{H} \otimes L^2(G)} \qquad \text{(by Lemma 18)} \\
&= \int_H \langle \{\hat{\delta}_h(y^* x) w\}(\xi), \eta \rangle_{\mathscr{H} \otimes L^2(G)}\, dh \\
&= \int_H \langle \{V(x \otimes \zeta)\}(h), \{V(y \otimes \eta)\}(h) \rangle_{\mathscr{H} \otimes L^2(G)}\, dh \\
&= \langle V(x \otimes \zeta), V(y \otimes \eta) \rangle_{L^2(H, \mathscr{H} \otimes L^2(G))}.
\end{aligned}$$

So V preserves the pre-inner products. Suppose $\gamma \in \mathscr{I}_H$, $x \otimes \zeta \in \mathscr{D} \otimes \mathscr{F}$, with $\zeta = w(\xi)$ and $\eta \in L^2(H, \mathscr{H} \otimes L^2(G))$. Then

$$\begin{aligned}
&\langle \{V(\Lambda \circ \Theta(\gamma))\}(x \otimes \zeta), \eta \rangle_{L^2(H, \mathscr{H} \otimes L^2(G))} \\
&= \int_H \langle \{V(\{[\theta_\gamma]\}(x) \otimes \zeta)\}(h), \eta(h) \rangle_{\mathscr{H} \otimes L^2(G)}\, dh
\end{aligned}$$

$$= \int_H \langle \{\hat{\delta}_h(\gamma \bullet x)\}(\zeta), \eta(h) \rangle_{\mathscr{H} \otimes L^2(G)}\, dh$$

$$= \int_H \int_H \langle \sqrt{\Delta r}\{\hat{\delta}_h(\gamma(r)\, \hat{\delta}_r(x))w\}(\xi), \eta(h) \rangle_{\mathscr{H} \otimes L^2(G)}\, dr\, dh$$

$$= \int_H \int_H \langle \{\hat{\delta}_h(\gamma(r))\}(\sqrt{\Delta r}\, \{V(x \otimes \zeta)\}(hr)), \eta(h) \rangle_{\mathscr{H} \otimes L^2(G)}\, dh\, dr$$

(since : $(h, r) \to \hat{\delta}_h(\gamma(r)\, \hat{\delta}_r(x))w$ is continuous with compact support)

$$= \int_H \int_H \langle \{\{\check{\imath}(\gamma(r))(\{1 \otimes \rho_H\}(r))\, V\}\, (x \otimes \zeta)\}(h), \eta(h) \rangle_{\mathscr{H} \otimes L^2(G)}\, dh\, dr$$

$$= \left\langle \left\{ \int_H \check{\imath}(\gamma(r))(\{1 \otimes \rho_H\}(r))\, dr V \right\} (x \otimes \zeta), \eta \right\rangle_{L^2(H, \mathscr{H} \otimes L^2(G))}$$

$$= \langle \{\{\check{\imath} \times (1 \otimes \rho_H)\}(\gamma)\, V\}(x \otimes \zeta), \eta \rangle_{L^2(H, \mathscr{H} \otimes L^2(G))}.$$

Hence $V(\Lambda \circ \Theta(\gamma)) = \{\check{\imath} \times (1 \otimes \rho_H)\}(\gamma) V$. So V extends to a unitary operator from Z onto $L^2(H, \mathscr{H} \otimes L^2(G))$ which intertwines the actions of $(A \times_\delta G) \times_{\hat{\delta}} H$. That is, $\check{\imath} \times (1 \otimes \rho_H)$ is unitarily equivalent to $\Lambda \circ \Theta$, as claimed. ∎

THEOREM 27. *Suppose δ is a non-degenerate coaction of a locally compact group G on a C^*-algebra A, H is a closed normal amenable subgroup of G, $\delta|$ is the restriction of δ to G/H, as in Lemma 4, and $\hat{\delta}$ is the dual action of G on $A \times_\delta G$. Then $(A \times_\delta G) \times_{\hat{\delta}} H$ is strongly Morita equivalent to $A \times_{\delta|} (G/H)$.*

Proof. By Theorem 19, $\mathscr{D}$ is a $\mathscr{D}_H$-rigged space, so by [21, Propositions 6.5 and 6.6] $K(\mathscr{D})$ is strongly Morita equivalent to $A \times_\delta (G/H)$. But by Propositions 26 and 7, respectively, $K(\mathscr{D})$ is isomorphic to $(A \times_\delta G) \times_{\hat{\delta}} H$, and $A \times_\delta (G/H)$ is isomorphic to $A \times_{\delta|} (G/H)$, establishing the theorem. ∎

Let $j_{A \times_\delta G}$ be as in Corollary 20. Let $i_{A \times_\delta G}$ be the natural inclusion of $A \times_\delta G$ in $M((A \times_\delta G) \times_{\hat{\delta}} H)$. Then $\Theta \circ i_{A \times_\delta G} = j_{A \times_\delta G}$, where Θ has been extended to a map from $M((A \times_\delta G) \times_{\hat{\delta}} H)$ to $M(K(\mathscr{D})) \subset L(\mathscr{D})$. Let $\alpha: L(\mathscr{D}) \to L(X)$ be the map determined by $\{\alpha(T)\}([x]) = [T(x)]$, for $x \in \mathscr{D}$, where $[y]$ is the equivalence class of y in X. Then α is a $*$-isomorphism which maps $K(\mathscr{D})$ onto $K(X)$.

Now applying [21, Theorem 6.29] to the Hermitian $A \times_\delta (G/H)$-rigged $A \times_\delta G$ module X, used in Section 4 to induce representations from $A \times_{\delta|} (G/H)$ to $A \times_\delta G$, we see that a representation μ of $A \times_\delta G$ on $\mathscr{Q}$ is induced (via X) from a representation ν of $A \times_{\delta|} (G/H)$ (more precisely, from the representation $v \circ \Gamma^{-1}$ of $A \times_\delta (G/H)$, where Γ is as in Proposition 7) if, and only if, there exists a representation ψ of $K(X)$ on $\mathcal{Q}$ such that

$$\{\mu(b)\}(\{\psi(T)\}(\xi)) = \{\psi(\alpha \circ j_{A \times_\delta G}(b) T)\}(\xi),$$

for all $b \in A \times_\delta G$, $T \in K(X)$, and $\xi \in \mathcal{Q}$. Recognizing $L(X)$ as $M(K(X))$ [8, Lemma 16] this is if, and only if, there exists a representation ψ of $K(X)$ on $\mathcal{Q}$ such that

$$\mu(b) = \psi(\alpha \circ j_{A \times_\delta G}(b)) \qquad \forall b \in A \times_\delta G,$$

where ψ has been extended to $M(K(X))$. This is if, and only if, there exists a representation ϕ $(= \psi \circ \alpha \circ \Theta)$ of $(A \times_\delta G) \times_{\hat{\delta}} H$ on $\mathcal{Q}$ such that

$$\mu(b) = \phi(i_{A \times_\delta G}(b)) \qquad \forall b \in A \times_\delta G,$$

but this is if, and only if, there exists a unitary representation U of H on $\mathcal{Q}$ such that (μ, U) is a covariant representation of $(A \times_\delta G, H, \hat{\delta})$, where $\hat{\delta}$ is the dual action of H on $A \times_\delta G$. So we have the following theorem:

THE IMPRIMITIVITY THEOREM 28. *A representation μ of $A \times_\delta G$ on $\mathcal{Q}$ is induced (via X) from a representation v of $A \times_{\delta|} (G/H)$ if, and only if, there exists a unitary representation U of H on $\mathcal{Q}$ such that (μ, U) is a covariant representation of $(A \times_\delta G, H, \hat{\delta})$.*

Note that we could have stated the imprimitivity theorem for representations induced via $\mathcal{D}$, since representations induced via $\mathcal{D}$ are unitarily equivalent to those induced via X.

6. THE CONTINUITY OF THE INDUCTION AND RESTRICTION PROCESSES

We refer to the induction and restriction processes introduced after Theorem 19 and in Proposition 5, respectively. Let Γ be the isomorphism of Proposition 7 and let $\mu = (\mu \circ \delta) \times (\mu \circ (1 \otimes M_G))$ [14, Theorem 3.7] be a representation of $A \times_\delta G$. Then

$$\begin{aligned} \operatorname{res}^{A \times_\delta G}_{A \times_{\delta|} (G/H)} \mu &= \operatorname{res}^{A \times_\delta G}_{A \times_{\delta|} (G/H)} ((\mu \circ \delta) \times (\mu \circ (1 \otimes M_G))) \\ &= (\mu \circ \delta) \times (\mu \circ (1 \otimes M_G) \circ q) \\ &= (\mu \circ \Gamma \circ \delta|) \times (\mu \circ \Gamma \circ (1 \otimes M_{G/H})) \\ &= \mu \circ \Gamma, \end{aligned} \tag{13}$$

where μ has been extended to $M(A \times_\delta G) \supset A \times_\delta (G/H)$. Now we present a similar result for $\mathrm{ind}_{A \times_{\delta|} (G/H)}^{A \times_\delta G}$. Recall that X is a Hermitian $A \times_\delta (G/H)$-rigged $K(X)$ module [21, Proposition 6.14] and thus establishes the induction process

$$\mathrm{ind}_{A \times_{\delta|} (G/H)}^{K(X)}\colon \mathrm{Rep}(A \times_{\delta|} (G/H)) \to \mathrm{Rep}(K(X)).$$

We define the map

$$\mathrm{ind}_{A \times_{\delta|} (G/H)}^{(A \times_\delta G) \times_{\hat{\delta}} H}\colon \mathrm{Rep}(A \times_{\delta|} (G/H)) \to \mathrm{Rep}((A \times_\delta G) \times_{\hat{\delta}} H)$$

by

$$\mathrm{ind}_{A \times_{\delta|} (G/H)}^{(A \times_\delta G) \times_{\hat{\delta}} H} \nu = (\mathrm{ind}_{A \times_{\delta|} (G/H)}^{K(X)} \nu) \circ \alpha \circ \Theta.$$

Then by Rieffel's imprimitivity theorem [21, Theorem 6.29]

$$\begin{aligned} \mathrm{ind}_{A \times_{\delta|} (G/H)}^{A \times_\delta G} \nu &= (\mathrm{ind}_{A \times_{\delta|} (G/H)}^{K(X)} \nu) \circ \alpha \circ j_{A \times_\delta G} \\ &= (\mathrm{ind}_{A \times_{\delta|} (G/H)}^{(A \times_\delta G) \times_{\hat{\delta}} H} \nu) \circ i_{A \times_\delta G}. \end{aligned} \tag{14}$$

Suppose $P\colon A \to M(B)$ is a $*$-homomorphism. Then we can define a map

$$P^*\colon \mathrm{Ideals}(B) \to \mathrm{Ideals}(A) \qquad \text{by} \quad P^*(I) = \{a \in A\colon P(a)B \subset I\},$$

where I is an ideal of B. Green shows that P^* preserves intersections and is continuous with respect to the (inner) hull-kernel topology, as defined in [8] just before Proposition 9. The relevance of P^* lies in the fact that if ψ: $B \to B(\mathcal{Q})$ is a representation of B on $\mathcal{Q}$, then

$$\ker(\psi \circ P) = \{a \in A\colon P(a)B \subset I\} = P^*(\ker \psi).$$

If μ is a representation of $A \times_\delta G$, then by (13), $\mathrm{res}_{A \times_{\delta|} (G/H)}^{A \times_\delta G} \mu = \mu \circ \Gamma$. Hence $\ker(\mathrm{res}_{A \times_{\delta|} (G/H)}^{A \times_\delta G} \mu) = \Gamma^*(\ker \mu)$, which shows that weakly equivalent representations of $A \times_\delta G$ are mapped to weakly equivalent representations of $A \times_{\delta|} (G/H)$. Hence the restriction process can be considered to be a map on ideals and as such is continuous, since Γ^* is. If ν is a representation of $A \times_{\delta|} (G/H)$, then from (14)

$$\begin{aligned} \ker(\mathrm{ind}_{A \times_{\delta|} (G/H)}^{A \times_\delta G} \nu) &= i_{A \times_\delta G}^*(\ker(\mathrm{ind}_{A \times_{\delta|} (G/H)}^{(A \times_\delta G) \times_{\hat{\delta}} H} \nu)) \\ &= i_{A \times_\delta G}^*(h(\ker \nu)), \end{aligned}$$

where h: $\mathrm{Ideals}(A \times_\delta (G/H)) \to \mathrm{Ideals}((A \times_\delta G) \times_{\hat{\delta}} H)$ is the continuous bijection of [22, Theorem 3.1] determined by the equivalence bimodule X. This shows that the induction process $\mathrm{ind}_{A \times_{\delta|} (G/H)}^{A \times_\delta G}$ maps weakly equivalent representations to weakly equivalent representations and can thus be considered a map on ideals. As such it is clearly continuous since both h and $i^*_{A \times_\delta G}$ are. So summing up we have the following proposition.

PROPOSITION 29. *The maps*

$$\operatorname{ind}_{A \times_{\delta|}(G/H)}^{A \times_\delta G}: \text{Ideals}(A \times_{\delta|} (G/H)) \to \text{Ideals}(A \times_\delta G)$$
$$\operatorname{res}_{A \times_{\delta|}(G/H)}^{A \times_\delta G}: \text{Ideals}(A \times_\delta G) \to \text{Ideals}(A \times_{\delta|} (G/H))$$

are continuous with respect to the hull-kernel topologies.

Proposition 21 shows that this result is an extension of Gootman and Lazar's Theorem 3.8 [7]. Theirs is the case $H = G$.

7. PROPER ACTIONS

An action α of a locally compact group G on a C^*-algebra B is said to be proper is there exists a dense α-invariant $*$-subalgebra B_0 of B such that

(i) for any $a, b \in B_0$ the maps $:s \to (1/\sqrt{\Delta s})\, a\alpha_s(b^*)$ and $:s \to a\alpha_s(b^*)$ are in $L^1(G, B)$,

(ii) for all $a, b \in B_0$ there exists an element $\psi_{a,b} \in M(B_0)^\alpha$, where $M(B_0)^\alpha$ denotes the α-invariant elements of $M(B)$ which carry B_0 into itself, such that

$$c\psi_{a,b} = \int_G ca\alpha_s(b^*)\, ds \qquad \forall c \in B_0.$$

Proper actions, which are closely related to the integrable actions of [1, 2], were formulated by Rieffel [24]. Let B^α denote the closure, in $M(B)$, of the elements $\psi_{a,b}$, for $a, b \in B_0$. Then Rieffel [24] has shown that B^α is strongly Morita equivalent to a subalgebra E of the reduced crossed product $B \times_{\alpha,r} G$. If E is all of $B \times_{\alpha,r} G$ we will call α saturated.

THEOREM 30. *Suppose δ is a non-degenerate coaction of G on A and H is a closed normal amenable subgroup of G. Then the dual action $\hat{\delta}$ of H on $A \times_\delta G$ is proper and saturated.*

Proof. Our candidate for B_0 will be $\mathscr{D}$. Lemma 11(vi) and Theorem 12 show that $\mathscr{D}$ is a dense $\hat{\delta}$-invariant $*$-subalgebra of $A \times_\delta G$. Let $x, y \in \mathscr{D}$. By Lemma 18 the maps: $h \to (1/\sqrt{\Delta h})\, x\hat{\delta}_h(y^*)$ and: $h \to x\hat{\delta}_h(y^*)$ are continuous and compactly supported, and hence in $L^1(G, B)$. Letting Ψ be the map of Lemma 16, it is clear from Lemma 11(iv) and (v) that $\Psi(x^*y) \in M(\mathscr{D})^{\hat{\delta}}$. Also, by Lemma 18,

$$z\Psi(x^*y) = \int_H z\hat{\delta}_h(x^*y)\,dh \qquad \forall z \in \mathscr{D}.$$

Hence $\hat{\delta}$ is proper. Now $(A \times_\delta G)^{\hat{\delta}}$ is the closure of the linear span of the elements $\Psi(x^*y)$, which is the closure of $\mathscr{D}_H$, that is, $A \times_\delta (G/H)$. By Theorem 27, $(A \times_\delta G)^{\hat{\delta}}$ is strongly Morita equivalent to $(A \times_\delta G) \times_{\hat{\delta}} H$, so $\hat{\delta}$ is saturated. ∎

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